Adaptive Sampling Noise Mitigation Technique for Feedback-based Quantum Algorithms

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Abstract. Inspired by Lyapunov control techniques for quantum systems, feedback-based quantum algorithms have recently been proposed as alternatives to variational quantum algorithms for solving quadratic unconstrained binary optimization problems. These algorithms update the circuit parameters layer-wise through feedback from measuring the qubits in the previous layer to estimate expectations of certain observables. Therefore, the number of samples directly affects the algorithm's performance and may even cause divergence. In this work, we propose an adaptive technique to mitigate the sampling noise by adopting a switching control law in the design of the feedback-based algorithm. The proposed technique can lead to better performance and convergence properties. We show the robustness of our technique against sampling noise through an application for the maximum clique problem.

Keywords: FALQON · QLC · Sampling Noise Mitigation.

1 Introduction

Solving combinatorial optimization problems is one of the leading applications where noisy intermediate scale quantum (NISQ) devices are expected to show an advantage over classical algorithms. For NISQ devices, the leading algorithms that can fulfil these devices' requirements and are expected to show quantum advantage are the variational quantum algorithms (VQAs) [3]. VQAs have applications in quantum chemistry, error correction, quantum machine learning, and combinatorial optimization [13, 3, 14].

In [12, 11], the feedback-based algorithm for quantum optimization (FALQON) was proposed as an alternative to VQAs to solve quadratic unconstrained binary optimization (QUBO) problems. Unlike VQAs, FALQON avoids the classical optimization problem associated with VQAs. Instead, it updates the circuit parameters layer-wise through feedback from measuring the qubits in the previous layer to estimate expectations of certain observables. FALQON was also applied

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to find the ground state of Hamiltonian, specifically in the Fermi-Hubbard model [10]. It can also be used as a potential initialization technique for the parameters of the Quantum Approximate Optimization Algorithm (QAOA) [12].

In FALQON, the parameters of the circuit are calculated by estimating the expected values of some observables using finite number of samples, leading to noise in the estimation of the parameters. This noise will be fed directly to the next layer through the feedback law, affecting the algorithm's performance.

In this work, leveraging tools from quantum Lyapunov control (QLC) theory, we propose an alternative controller design that can mitigate the noise caused by the finite number of samples. Inspired by the switching controller design for QLC proposed in [9], we propose a switching control law that switches between the standard Lyapunov control law and the bang-bang control law. Unlike the open loop control problem of quantum systems considered in [9], FALQON requires estimating expected values of observables via quantum measurements, the number which we strive to reduce by proposing an adaptive sampling technique with the switching bang-bang and standard Lyapunov control feedback law.

We apply FALQON to the maximum clique problem (MCP) with our proposed approach. Through simulations, we show that our proposal is robust against sampling noise and can perform better than the standard Lyapunov technique employed in FALQON with the same number of samples.

The remainder of the paper is structured as follows. In Section 2, we review QLC and FALQON. In Section 3, we introduce our adaptive sampling noise mitigation technique. In Section 4, we investigate the robustness of our approach against sampling noise through application to MCP. Finally, we give a conclusion in Section 5.

2 Preliminaries

In this section, we provide an overview of FALQON [12, 11] for finding ground states and solving QUBO problems. We start by reviewing QLC and subsequently establish its connection to FALQON.

2.1 Quantum Lyapunov Control

Let us consider the Hilbert space $\mathcal{H} = \mathbb{C}^L$ with associated orthonormal basis $\mathcal{B} = \{|n\rangle\}_{n\in\{0,\ldots,L-1\}}$ and the set of quantum states given by $\mathcal{Y} = \{|\psi\rangle \in \mathbb{C}^L : \langle\psi|\psi\rangle = \| \, |\psi\rangle \, \|^2 = 1\}$. In the following, all operators will be represented on the $\mathcal{B}$ basis. Consider a quantum system whose dynamics are governed by the controlled time-dependent Schrödinger equation

$$i|\dot{\psi}(t)\rangle = (H_0 + u(t)H_1)|\psi(t)\rangle. \tag{1}$$

where the Planck constant $\hbar$ is normalized to 1, $u(t)$ is the control input and H_0 and H_1 are the drift and control Hamiltonian, respectively. In this work, both H_0 and H_1 are assumed to be time-independent and non-commuting,

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i.e., $[H_0, H_1] \neq 0$. The main objective here is to find a control law in a feedback form, $u(|\psi(t)\rangle)$, that guarantees the convergence of the quantum system (1), from any initial state to the ground state of the Hamiltonian H_0, i.e., the state $|\psi_g\rangle = \text{argmin}_{|\psi\rangle\in\mathcal{H}} \langle\psi| H_0 |\psi\rangle$. Consider a Lyapunov function of the form $V(|\psi\rangle) = \langle\psi| H_0 |\psi\rangle$, whose derivative along the trajectories of system (1) is given by $\dot{V}(|\psi(t)\rangle) = \langle\psi(t)| \, \mathrm{i}[H_1, H_0] \, |\psi(t)\rangle \, u(t)$. Hence, designing $u(t)$ as

$$u(t) = -Kf(\langle\psi(t)| \, \mathrm{i}[H_1, H_0] \, |\psi(t)\rangle), \tag{2}$$

where $K > 0$ and f is a continuous function satisfying $f(0) = 0$ and $xf(x) > 0$, for all $x \neq 0$, will ensure that $\dot{V} \leq 0$. Applying the controller (2), under some assumptions [12, 6], guarantees asymptotic convergence to the ground state $|\psi_g\rangle$.

2.2 Feedback-Based Quantum Optimization Algorithm

The quantum evolution propagator $U(t)$ associated to (1) is $U(t) = \tau e^{-i\int_0^t H(t')dt'}$, where τ is the time-ordering operator. By breaking it into p number of piecewise constant time intervals of length Δt, we get $U(T, 0) \approx \prod_{k=1}^{p} e^{-iH(k\Delta t)\Delta t}$, where the time step Δt is chosen to be small enough such that H(t) is approximately constant over the interval Δt. This can be simplified using Trotterization as $U(T, 0) \approx \prod_{k=1}^{p} e^{-iu(k\Delta t)H_1\Delta t} e^{-iH_0\Delta t}$. Hence, we get a digitized formulation of the evolution in the form

$$|\psi_p\rangle = \prod_{k=1}^{p} e^{-iu(k\Delta t)H_1\Delta t} e^{-iH_0\Delta t} |\psi_0\rangle = \prod_{k=1}^{p} U_1(u_k) U_0 |\psi_0\rangle, \tag{3}$$

where $u_k = u(k\Delta t)$, $|\psi_k\rangle = |\psi(k\Delta t)\rangle$, $U_0 = e^{-iH_0\Delta t}$, $U_1(u_k) = e^{-iu(k\Delta t)H_1\Delta t}$.

For a drift Hamiltonian H_0 specified as a sum of Pauli strings as $H_0 = \sum_{q=1}^{N_0} c_q O_q$, where c_q's are real scalar coefficients, N_0 is given as a polynomial function of the number of qubits and $O_q = O_{q,1} \otimes O_{q,2} \otimes ... \otimes O_{q,n}$ with $O_{q,d} \in \{I, X, Y, Z\}$, the unitary U_0 can be efficiently implemented as a quantum circuit. Similarly, to be able to implement the operator U_1 as a quantum circuit efficiently, the Hamiltonian H_1 should be designed as $H_1 = \sum_{q=1}^{N_1} \hat{c}_q \hat{O}_q$.

The quantum circuit that implements $U(T, 0)$ simulates the propagator $U(t)$, where choosing Δt sufficiently small, can guarantee that $\dot{V} \leq 0$ [12]. The following feedback law is adopted:

$$u_{k+1} = -Kf(\langle\psi_k| \, \mathrm{i}[H_1, H_0] \, |\psi_k\rangle) = -Kf(\alpha_k), \tag{4}$$

where $\alpha_k = \langle\psi_k| \, \mathrm{i}[H_1, H_0] \, |\psi_k\rangle$. This is a discrete version of the controller (2). In [12], the function $f(\cdot)$ is chosen to be the identity function i.e. $f(\alpha) = \alpha$, and the gain K is set to 1. This particular choice of the function $f(\cdot)$ is known as the standard Lyapunov control law. Hence, we obtain the following controller:

$$u_{k+1} = -\alpha_k \tag{5}$$

The implementation of this quantum algorithm follows the algorithmic steps outlined below. The initial step involves seeding the procedure with an initial

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value for u_1 and setting a value for the time step Δt. Subsequently, a group of qubits is initialized to an easy-to-prepare initial state $|\psi_0\rangle$, and a single circuit layer is applied to prepare the state $|\psi_1\rangle$. The controller for the next layer of the quantum circuit u_2 is estimated on the quantum computer. To estimate the controller, we expand α in terms of Pauli strings as follows:

$$\alpha_k = \langle\psi_k| \, \mathrm{i}[H_1, H_0] \, |\psi_k\rangle = \sum_{q=1}^{N} \bar{c}_q \langle P_q\rangle_k, \tag{6}$$

where we use the notation $\langle P_q\rangle_k = \langle\psi_k| \, P_q \, |\psi_k\rangle$, P_q is a Pauli string and N is the number of Pauli strings. Note that N depends on N_0 and N_1, and since N_0 and N_1 are given as polynomial functions of the number of qubits, then N is also a polynomial function of the number of qubits. A new layer is then added to the circuit, and this sequence is iteratively followed for a depth of p layers. The dynamically designed quantum circuit $\prod_{k=1}^{p} U_1(u_k)U_0$ along its parameters $\{u_k\}_{k=1}^{p}$ constitutes the output of the algorithm. This output can effectively approximate the ground state of the Hamiltonian H_0.

Algorithm 1 FALQON [12]

Input: H_0, H_1, Δt, p, $|\psi_0\rangle$
Output: circuit parameters $\{u_k\}_{k=1}^{p}$
1: Set $u_1 = 0$
2: **Repeat** at every step $k = 1, 2, 3, \ldots, p-1$
3: Prepare the initial state $|\psi_0\rangle$
4: Prepare the state $|\psi_k\rangle = \prod_{l=1}^{k} U_1(u_k)U_0 \, |\psi_0\rangle$
5: Estimate α and calculate the controller u_{k+1} using (5)
6: **Until** $k = p$

3 Proposed Design of the Controller

In this section, we present our approach for modifying FALQON. We propose an alternative controller design using a switching control law.

In practice, for a finite number of samples $M < \infty$, we have a noisy estimate of the expectation $\tilde{\alpha}$. Hence the calculated controller becomes $\tilde{u}_{k+1} = -K\tilde{\alpha}_k$, where $\tilde{\alpha}_k$ is a noisy estimate of α_k. As the previous section shows, this controller will be directly fed to the next layer. Therefore, it will directly affect the performance of FALQON. To tackle this problem, we propose using a switching control law instead of the standard Lyapunov control law, where the control switches between a bang-bang controller and a standard Lyapunov control law. The bang-bang control switches between two states based on the sign of $\tilde{\alpha}$ and discards the noisy estimate of the expectation, thus mitigating its effect on the algorithm. In this work, we adopt the following switching control law. By defining $\epsilon > 0$ to be the additive error in the estimation of the expectation α_k, the

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noisy estimate of the controller ($M < \infty$) is given as, with probability $(1 - \delta)$:

$$\tilde{u}_{k+1} = -K\,\mathrm{sat}(\tilde{\alpha}_k) = \begin{cases} -W & \tilde{\alpha}_k - \epsilon > \phi \\ W & \tilde{\alpha}_k + \epsilon < -\phi \\ -K\tilde{\alpha}_k & \text{otherwise} \end{cases} \tag{7}$$

where $K > 0$ is the controller gain, $\phi > 0$ is a parameter to adjust the switching band, $W > 0$ and $0 < \delta < 1$. Note that, with probability $1 - \delta$, the estimated controller is totally equivalent to the noiseless controller in the region where the bang-bang controller is activated, i.e. in the region $|\tilde{\alpha}_k| > \phi + \epsilon$. Hence, we achieve sampling noise mitigation in this region by controller design.

We now quantify the error bound ϵ. From (6), to calculate $\alpha = \alpha_k$, we need to estimate N expectations of Pauli strings. For this case, performing a single-shot measurement of the qubits in a single circuit instance is the same as sampling an element from a distribution over $\{-1, 1\}$ with an expected value denoted as $\langle P_q \rangle_k$. To quantify the difference between the actual expectation and the estimated values, we employ Hoeffding's inequality. Hoeffding's inequality states that when provided with a sample of M independent and bounded random variables $\{X_i\}_{i=1}^M$ drawn from any distribution where $X_i \in [-\beta, \beta]$, the difference between the empirical expected value $\tilde{X}$ and the actual expected value satisfies the subsequent inequality:

$$\Pr\left(|\tilde{X} - \langle X \rangle| \geq \epsilon\right) \leq 2\exp\left(-\frac{M\epsilon^2}{2\beta}\right) =: \delta, \tag{8}$$

which implies that through a number of samples $M \geq 2\log(2/\delta)/\epsilon^2$, the expectation value $\langle P_q \rangle$ can be estimated within a precision of ϵ with probability $1 - \delta$ [2]. Since $\alpha = \sum_{q=1}^N \bar{c}_q \langle P_q \rangle$, we consider (8) substituting $\epsilon/(|\bar{c}_q|N)$ for ϵ and δ/N for δ. Namely, through $M_q \geq 2\log(2N/\delta)\bar{c}_q^2 N^2/\epsilon^2$ samples the expectation value $\langle P_q \rangle$ can be estimated within a precision of $\epsilon/(|\bar{c}_q|N)$ with probability $1 - \delta/N$, and therefore α can be estimated within a precision of ϵ with probability larger than or equal to $1 - \delta$. In fact,

$$P(|\tilde{\alpha} - \alpha| \geq \epsilon) = P(|\sum_q \bar{c}_q(\langle \tilde{P}_q \rangle - \langle P_q \rangle)| \geq \epsilon) \leq P(\sum_q |\bar{c}_q| \cdot |\langle \tilde{P}_q \rangle - \langle P_q \rangle| \geq \epsilon)$$

$$\leq P(\vee_q \{|\bar{c}_q| \cdot |\langle \tilde{P}_q \rangle - \langle P_q \rangle| \geq \frac{\epsilon}{N}\}) \leq \sum_q P(|\langle \tilde{P}_q \rangle - \langle P_q \rangle| \geq \frac{\epsilon}{|\bar{c}_q|N}) \leq \sum_q \frac{\delta}{N} = \delta.$$

Hence, by choosing δ and $M = \max_q M_q$, we can find the error bound on α as $\epsilon = \max_q |\bar{c}_q| \cdot N\sqrt{2\log(2N/\delta)/M}$. Based on this, we can calculate the switching control law defined by (7). From this inequality, we note that by increasing the number of samples, we can decrease the value of the error bound and, hence, increase the regions in which the bang-bang controller will be activated. In this way, we can adopt an adaptive number of samples, where we start with a small number of samples and check the condition $|\tilde{\alpha}_k| > \phi + \epsilon$ if it is not satisfied, we increase the number of samples to mM, where $m > 1$ and repeat till we satisfy

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the condition or reach a maximum number of samples $M_{\max}$. Simulation results suggest that this bound is conservative. To address this, we introduce a parameter K_e to adjust the bound, resulting in $\tilde{\alpha}_k \pm K_e \epsilon$ in the controller design. Note that better bounds could be achieved by adopting advanced estimation methods of the expectation α such as adaptive informational complete measurements [7] and classical shadows of quantum states [8]. We defer the analysis and comparison of these methods to future work. The implementation steps of our approach are outlined in Algorithm 2. This algorithm serves as a subroutine for FALQON, replacing step 5 in Algorithm 1.

Algorithm 2 Switching control law with adaptive number of samples

```
Input: |ψ_k⟩, δ, M, m, M_max, φ, K_e
Output: circuit parameter of the next layer u_{k+1}
Prepare the state |ψ_k⟩ = ∏_{l=1}^{k} U_1(u_k) U_0 |ψ_0⟩
while M ≤ M_max
  Calculate ε using ε = max_q |c̄_q| · N sqrt(2log(2N/δ)/M)
  Estimate α̃_k on the quantum computer using (6)
  if |α̃_k| > φ + K_e ε
      Calculate the controller u_{k+1} using (7)
  else
      M ← mM
end while
Calculate the controller u_{k+1} using (7)
```

4 Application to the Maximum Clique Problem

We apply our proposed approach to MCP, known to be an NP-complete optimization problem [4]. For MCP, we are given a graph $G = (V, E)$ where V is the set of vertices and E is the set of edges. A clique refers to a group of vertices that form a fully connected subgraph, where every pair of vertices within this group is connected by an edge in G. The size of the clique corresponds to the number of vertices it contains. The objective of MCP is to identify a clique within G that consists of the greatest possible number of vertices. In [4], MCP is formulated as the following QUBO minimization problem:

$$\min_{x \in \{0,1\}^n} -A \sum_{i \in V(G)} x_i + B \sum_{(i,j) \in E(G^c)} x_i x_j, \tag{9}$$

where B/A is chosen to be sufficiently large. To apply FALQON to this problem, we first map it into the following Ising Hamiltonian [4]:

$$H_0 = \sum_{i \in V(G)} Z_i + 3 \sum_{(i,j) \in E(G^c)} (Z_i Z_j - Z_i - Z_j) \tag{10}$$

where we set $A = 1$, $B = 3$. For numerical simulation, we consider the instance of MCP with $V = \{0, 1, 2, 3, 4\}$ and $E = \{\{0, 1\}, \{0, 2\}, \{1, 4\}, \{1, 2\}, \{2, 3\}\}$. For

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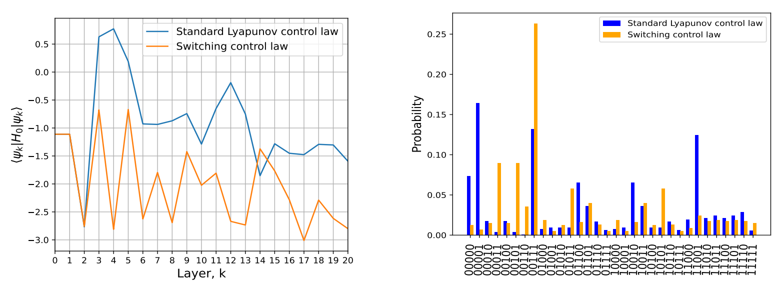
more details on implementing FALQON for MCP, see [15]. To execute FALQON, we use the Qiskit Aer quantum simulator [1]. We design the cost Hamiltonian as $H_1 = \sum_{i=1}^{5} X_i$ and the initial state as the equal superposition state $|\psi_0\rangle = |+\rangle^{\otimes 5}$. We set the time step to be $\Delta t = 0.18$ and the number of samples to be $M = M_{\max} = 5$. We run the algorithm using the standard Lyapunov technique and our approach for circuit depth of 20 layers. The switching control law parameters are set to be $W = 3$, $K = 2$, $\phi = 0.02$, $K_e = 0.035$ and $\delta = 0.2$. The values of W and K are chosen to match the standard Lyapunov control while ϕ, K_e and δ are adjusted to increase the band where the bang-bang controller is activated. The results in Figure 1 show that for a small number of samples $M = 5$, our proposed algorithm has better convergence to the ground state, which encodes the optimal solution to the problem $|00111\rangle$ with optimal value -4.75.

In addition, we run the algorithm for 20 realizations and plot the mean and the corresponding standard deviation. The results are shown in Figure 2. From Figure 2, it is seen that for our proposed approach using the switching control



(a) Lyapunov function $\langle\psi_k| H_0 |\psi_k\rangle$ plotted versus the layer k.

(b) The measurements' results of the final state for both approaches.

Fig. 1: Simulation results of one run of FALQON applied to the MCP instance using the standard Lyapunov control law and the proposed switching control law for a number of samples $M = 5$.

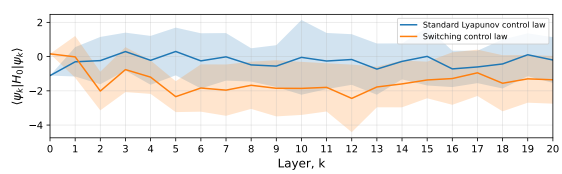


Fig. 2: Simulation results of applying FALQON to the MCP instance. FALQON is executed for 20 realizations and a number of samples $M = 5$, comparing the standard Lyapunov method with the proposed approach. The layer k is plotted versus the mean trajectory (solid line) and the corresponding standard deviation (shaded area) of the Lyapunov function $V_k = \langle\psi_k| H_0 |\psi_k\rangle$ for both approaches.

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law, the mean of the realizations decreases with the layer depth increase, while it fails for the standard approach. We note that for larger values of M, the Lyapunov function has better convergence for all realizations with increasing depth. However, both techniques have approximately similar performance.

5 Conclusion and Future Work

In this work, we proposed an adaptive sampling noise mitigation technique. Simulation results show that our approach is robust against sampling noise in an example of MCP. Our future work will focus on adapting this adaptive sampling noise mitigation technique to other quantum algorithms that depend on quantum control theory, such as the quantum imaginary-time evolution algorithm [5].

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