A Numerical Feed-Forward Scheme for the Augmented Kalman Filter

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Abstract. In this paper we present a numerical feed-forward strategy for the Augmented Kalman Filter and show its application to a diffusion-dominated inverse problem: heat source reconstruction from boundary measurements. The method is applicable in general to forcing term estimation in lumped and distributed parameters models and gives a significant contribution where, in industry and science, probing signals are used through a diffusive material-body to estimate its localized internal properties in a non-destructive test, like in ultrasound or thermographic inspection.

Keywords: Kalman Filter · forcing term estimation · feed-forward control · inverse heat transfer problems.

1 Introduction

There is an increasing interest, in applications, at estimating forcing terms of various nature (e.g. mechanical, thermal, etc.) from a relatively small set of data obtained by measurements of physical quantities (e.g. displacements, temperatures, etc.) that are assumed to be sufficiently informative of the effect of these forcing terms on the considered real system. This estimate represents a virtual measurement of the (physically) unmeasurable forcing term, performed by an algorithm which is usually called a *soft sensor* [4].

If the real system can be represented by a physico-mathematical model, then the inverse problem of estimating the (input) forcing term from output measurements must be solved by taking into account the model, the ill-posedness of the problem, and the uncertainty about both the data measured and the model's parameters. These aspects call for an interdisciplinary approach and for methods that are able to tackle with both the deterministic and the stochastic aspects of the problem. For this reason, in this field of applications, Bayesian methods have emerged as a main approach and a well known achievement in this direction, widely used in applications, is the Kalman Filter (KF) [11] [13].

To solve the problem of estimating the input forcing term, the reference model for the KF is often configured by augmenting the state vector to include the dynamics of the forcing term (input), and suppose the system be driven by white noise. In this way, the forcing term estimation problem becomes a state estimation problem, where the reference model represents the dynamics of two

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subsystems, i.e. the forcing term generator and the real system, and also of their interaction. This is usually uni-directional, i.e. the system's state does not affect the forcing term. This case is paradigmatic for the so-called Augmented Kalman Filter (AKF): the outputs are taken from the original state variables, i.e. the augmented state variables are unmeasurable, and the original variables does not affect the augmented variables.

In this paper we will consider this precise setting, where the reference model of the AKF represents explicitly the forward dependence between the forcing term and the system's state, while the AKF gain tries to estimate implicitly the backward dependence between them, in its effort to estimate the forcing term from the output prediction error. Our aim is to introduce an explicit representation of this backward dependence and feed it forward into the AKF state predictor. To do so, we compute a simulation of the inverse model and we will show and compare two possible strategies for using it to add a feed-forward term to the proportional control of the estimation error made by the standard AKF formulation. In this paper we will face diffusion dominated problems with distributed forcing terms, a difficult case for existing methods, as we will see.

In the literature, for source estimation in mechanical applications see e.g. Lourens [16] where the Augmented Kalman Filter (AKF) has been introduced for the purpose of mechanical load estimation, and the introduced covariance model that describes the forcing term modelling error is used as a regularizer in the force estimation problem, and precisely the diagonal elements of the covariance matrix are tuned as regularization parameters. Indeed, this covariance matrix can be seen also as the weighting matrix of a least-squares problem [13] and this is actually one way we formulate our feed-forward action in the KF, although not the best, as we will see. In [16] there is also an experimental comparison with other combined deterministic-stochastic techniques to force estimation problems. In Neats [17] there is an analysis of the stability of the AKF, which shows that there are common measurement configurations that exhibit a drift in the state estimates, due to unobservability issues and propose to add dummy-measurements; in [19] the sparse constraint is used to solve the drift problem in AKF with more generality. See [15] for a recent, general survey on mechanical load estimation techniques in both frequency and time domains.

For thermal applications, like the one here considered as a model problem, the application of the Kalman Filter is more problematic. The Augmented Kalman Filter is used e.g. in Qi et al [18], where there is also a comparison and references with another approach: KF-RLSE, i.e. the combination of a Kalman Filter and a Recursive Least-Squares Estimator of the residuals obtained from the KF. In [18] the heat flux (forcing term) is simply a scalar term, not a distributed field like is set in this paper. Another approach is the Optimal two-stage Kalman filter (OTSKF) and is preferred to the AKF because of its poor ability for the simultaneous estimation of spatio-temporal heat flux and temperature field. In this paper we show that a feed-forward scheme improves dramatically this ability. Actually, some years ago the two-stage Kalman filter had been proposed in [14] as an efficient implementation of the Augmented Kalman Filter and, in principle,

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the feed-forward scheme could be applied also for this formulation, if preferred. Finally, we mention a fundamental work by Gillijns e De Moor [9], that is not applicable here, anyway, since it requires that the number of measurement points be greater or equal to the forcing term nodes/variables.

In this paper we assume that the input generator and the real system are modeled adequately, since our scope is on finding a correct and efficient formulation for the feed-forward action. First of all, we must legitimate this choice: note that the standard AKF operates a proportional control over the state estimation error and it is well known that in problems where the input-state dynamics are slow this results in poor performances. There are variants where a proportional-integral actions [3] [2] [8] is performed, at the cost of doubling the number of state variables, or a *smoothing* is done through post-processing [1]. Actually, a feed-forward [12] approach is a consolidated good practice in control theory in general and a combination of feed-forward with feedback to eliminate steady-state errors is mostly used: feed-forward is used for tracking capabilities, and feedback for steady-state accuracy.

The paper is organized as follows: in Sec. 2 we set model problem and AKF formulation; in sec. 3 the feed-forward strategy is presented; in sec. 4 we show some numerical results and a Discussion and Conclusions section ends the paper.

2 Problem settings and Kalman Filter estimation of the forcing term

As a model problem of a diffusive process, let us consider the heat equation

$$\begin{cases} \rho C\, \partial_t T_{(f_\vartheta)} = \kappa\, \Delta T_{(f_\vartheta)} + f_\vartheta, & in\ D_c^{(0)} \times [0, t_f] \\ \kappa\, \nabla T_{(f_\vartheta)} \cdot \mathbf{n}_S = q(t), & on\ S \times [0, t_f] \\ \kappa\, \nabla T_{(f_\vartheta)} \cdot \mathbf{n} = 0, & on\ \delta D_c^{(0)}/S \times [0, t_f] \\ T_{(f_\vartheta)}(0, \cdot) = T_0(\cdot), & in\ D_c^{(0)}. \end{cases} \tag{1}$$

and suppose that the heat source term f_ϑ is an unknown function, in general, except that it is assumed different from zero only in a few disconnected regions of compact support. This is a common situation in many applications.

The aim of this paper is to estimate f_ϑ from a limited number of temperature measurements $\tilde{T}_{(f_\vartheta)}$, typically taken at the boundary. The restriction to a 2D problem is only for simplicity, the method we devise can be used in higher dimensions. The estimate of f_ϑ can be seen as an indirect measurement of f_ϑ from physical temperature measurements and we do this by exploiting the combination of the physico-mathematical model 1 and an abstract, data-driven model to describe f_ϑ. Therefore, the resulting method may be called a *physics-aware soft-sensor* [4].

Let us consider problem (1), discretized in space using the Finite Element Method (FEM) with Lagrangian elements P1, i.e. first-degree piecewise polynomials, and in time with the implicit Euler method, obtaining at iteration k:

$$M \frac{\tilde{T}_k - \tilde{T}_{k-1}}{d_t} = K\tilde{T}_k + f_k \qquad \Rightarrow \qquad (I - d_t\ M^{-1}K)\tilde{T}_k = \tilde{T}_{k-1} + d_t\ M^{-1} f_k$$

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where $M \in R^{n\times n}$ and $K \in R^{n\times n}$ are the mass and stiffness matrices of the FEM discretization, d_t is the time step chosen in the time discretization and f_k is the heat source at time t_k. If we knew all the values of $T_{(f_\vartheta)}$ then the computation of the unknowns f_ϑ would be a simple algebraic reconstruction. Since we can measure only a few components of $T_{(f_\vartheta)}$, we can reformulate this model as a state-space dynamical system with the aim of estimating its state from output measurements. Let us consider the following state-space discrete model in physical coordinates:

$$\begin{aligned} x_m(k+1) &= A_m x_m(k) + B_m u_m(k) + v_m(k) \\ y_m(k) &= C_m x_m(k) + w(k) \end{aligned} \tag{2}$$

where $x_m(k) = \tilde{T}_k$, $u_m(k) = f_k$, $A_m = (I - d_t M^{-1}K)^{-1}$, $B_m = A_m d_t M^{-1} = (I - d_t M^{-1}K)^{-1} d_t M^{-1}$, C_m is a matrix built with the rows of the identity matrix corresponding to measured nodes, $v_m(k)$ is a stochastic term for model error and $w(k)$ for measurement error, both supposed Gaussian noise [11]. Now, we augment the state of (2) by adding a model for the forcing term f_k:

$$x(t) = \begin{bmatrix} x_f \\ x_m \end{bmatrix} \quad , \quad u(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

where $x(t)$ is the *augmented state vector*, $x_f = f_k$, $u(t)$ the new input vector, which can be now omitted, $y(t)$ the output vector and:

$$A = \begin{bmatrix} A_f & 0 \\ Z & A_m \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 0 & C_m \end{bmatrix} \quad , \tag{4}$$

where $A_f = I$, $Z = B_m C_u$, $C_u = I$ and the augmented state-space model is:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + v(k) \\ y(k) &= Cx(k) + w(k) \end{aligned} \tag{5}$$

Now, to estimate the state vector it is a common choice to adopt a Kalman Filter, that we recall here in its one-step version [13]:

$$P(k) = \left[\left(Q(k-1) + A(k-1)P(k-1)A(k-1)^T \right)^{-1} + C^T R^{-1} C \right]^{-1} \tag{6}$$

$$e_{outpred} = [C\,(A(k-1)\ \hat{x}(k-1) + B\ u(k-1)) - \bar{y}(k)] \tag{7}$$

$$\delta\hat{x}(k) = -P(k)\ C^T R^{-1} e_{outpred} \tag{8}$$

$$\hat{x}(k) = A(k-1)\ \hat{x}(k-1) + B\ u(k-1) + \delta\hat{x}(k) \tag{9}$$

where the state vector $x(k)$ is the concatenation of the temperatures $T_{(f_\vartheta)}$ computed at the mesh nodes and the source term f_ϑ at the same nodes

$$x(k) = \begin{bmatrix} f_\vartheta(k) \\ T_{(f_\vartheta)}(k) \end{bmatrix} \quad , \tag{10}$$

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while the vector $y(k)$ contains the measured temperatures. In the problem at hand, the reference model for the Kalman Filter is the union of a constant model for the heating source (this is the usual choice made in the literature for this kind of problem) and the FE discretization of (1), and we set the covariance matrix of the model error as

$$Q(k) = \begin{bmatrix} Q_f(k) & Q_{mf}^T(k) \\ Q_{mf}(k) & Q_m(k) \end{bmatrix} \tag{11}$$

where $Q_f(k) = \sigma^2_{Q_f}\ I$, $Q_m(k) = \sigma^2_{Q_m}\ I$ and $Q_{mf}(k) = \sigma^2_{Q_{mf}} I$ in general, since we don't know the location of the forcing term. Then, set the initial covariance matrix of the state-estimation error as

$$P(0) = \begin{bmatrix} P_f(0) & P_{mf}(0)^T \\ P_{mf}(0) & P_m(0) \end{bmatrix} \tag{12}$$

where $P_f(0) = \sigma^2_{P_f} I$ and $P_m(0) = \sigma^2_{P_m} I$.

From equation (8) it is evident that the KF operates a proportional feedback action on the (scaled) output prediction error, with a gain

$$G_{KF}(k) = -P(k)\ C^T = \begin{bmatrix} P_{mf}(k)^T C_m^T \\ P_m(k) C_m^T \end{bmatrix} \tag{13}$$

and, since the product $P_{mf}(k)^T C_m^T$ here simply corresponds to a selection of columns of the matrix $P_{mf}(k)^T$, the proportional action made on the x_f variables can be understood better if we compute the recursive relation for $P_{mf}(k)$. At the limit $R = \infty$, from (6) we would have

$$\begin{aligned} P_{mf}^T(k) &= \left(\sigma^2_{P_f}\ Z + A_m P_{mf}(k-1)\right)^T + Q_{mf}^T(k) \\ &= \left(\sigma^2_{P_f}\ d_t A_m M^{-1} + A_m P_{mf}(k-1)\right)^T + Q_{mf}^T(k) \end{aligned} \tag{14}$$

Note that this would bring to $\delta\hat{x}(k) = 0$ since $C^T R^{-1} = 0$. Anyway, with realistic values of R, there is a rank-n_y modification of the inverse of $P(k)$ whose influence on $P_{mf}^T(k)$ is secondary in this analysis.Then, since A_m is the discretization of a diffusive operator, it will simply diffuse the output prediction error $e_{outpred}$ (7) to compute the state estimate update $\delta\hat{x}(k)$ (8), the matrix Q_{mf} cannot help, since we don't know the location of the sourcing term, and the poor result is depicted in sec. 4.1, Figure 2.

In general, it is well known that this proportional feedback may have strong limitations, e.g. for systems with a substantial inertia like thermal systems have, and for this reason a few algorithmic extensions have been developed in the literature, like e.g. a proportional-integral formulation [2], which requires a doubling of the state-variables, or like some additional post-processing (precisely, *smoothing* [1]) of the KF predictions. These are yet general improvements, that do not exploit the reference model as it could be done in the model problem here considered. For this reason we add instead a feed-forward action, described in the

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next section. At our knowledge, there is no contribution in the literature about introducing a feed-forward action into the Kalman Filter algorithm. In this paper, we do this for the Augmented Kalman Filter, with the assumption that the variable augmentation is done to describe the dynamics of an input forcing term. Therefore, there is a precise modelling assumption underlying this feed-forward scheme. In principle, the numerical scheme we propose may be used in general Kalman filtering, but a more general supporting assumption is not clear, at this moment, so we omit to discuss it.

3 A Feed-Forward strategy for the Augmented Kalman Filter

The unknown forcing term can be interpreted as a deterministic load disturbance, whose behaviour can be at least partially known. This would call for a feed-forward mechanism added to the proportional feedback activity performed by the Kalman Filter but, since we cannot measure the forcing term, a standard feed-forward would be a trial-and-error with no value added, since it should assume to know the quantity that we are estimating. Indeed, our idea is to exploit the maximum principle: if the output prediction error has a mean value different from zero we attribute this fact to an internal heat source and use it to drive the feed-forward action, as it is described in the following subsections.

3.1 Modeling the feed-forward action

Since we should obtain the feed-forward reference signal from the output prediction error (7), the most straightforward approach is to embed the feed-forward action into the Kalman gain (13). The idea is to include the relation between the output prediction error and the load disturbance (i.e. the inverse model) inside the covariance matrix $Q(k)$, as a deterministic least-squares weighting, following the interpretation of [13] of the KF. note that $Q_{mf}(k)$ in (14) translates the effect of a temperature error to an error in the forcing term, through the Kalman gain computation (13). Depending on how be build $Q_{mf}(k)$, this can be interpreted in different ways. Here we set each i-th row of $Q_{mf}(k)$ as the response of the model at a unit forcing term concentrated at the node i, after a number of time-steps that depends on the distance of the node from the measured boundary segment. Therefore, when the KF inverts the covariance matrix, this will translate a variation of the temperatures at the border into a variation of the forcing term additional to that operated by the standard Kalman Filter with $Q_{mf}(k) = 0$ (this would be a typical choice, since $Q_{mf}(k)$ classically expresses the mutual covariance between the model error of forcing term variables and temperature field variables, which can be assumed to be zero).

The action created by $Q_{mf}(k)$ is called *feed-forward* because it depends on the model between the load disturbance and the process output, and precisely on the inversion of that model; in this way, the apriori estimate of the forcing term is substituted by the output prediction error, under the assumption that a disturbance load is the main cause of it, as it is the case for the model problem

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here considered. Now the question becomes how to tune this feed-forward action. Note that $Q_f(k)$, $Q_m(k)$, $P_f(0)$, $P_m(0)$ and $P_{mf}(0)$ are tuned independently, e.g. following a best practice in Kalman filtering (see e.g. [11]). Let us consider equations (8)-(9) and the augmented system's matrices (5). We have

$$\begin{bmatrix} \hat{x}(k) - \hat{x}(k-1) \\ \vdots \end{bmatrix} = \begin{bmatrix} -P_{mf}(k)^T C_m^T R^{-1} \left[C_m A_m \hat{x}_m(k-1) + C_m Z \hat{x}_f(k-1) - \bar{y}(k) \right] \\ \vdots \end{bmatrix} \tag{15}$$

Now, let us do n_f simulations where $\hat{x}_m(0) = x_m(0)$, $\hat{x}_f(0) = 0$ and $x_f(0)$ is zero everywhere except a single value which is equal to $L = 100$ (i.e. a pointwise forcing term), corresponding to a different mesh node at each experiment. In this way, from the first n_u equations of (15), we obtain a linear system with multiple right-hand side $M_{sl} = M_{PC} M_{ope}$ where each column of $M_{sl} \in R^{n_u \times n_f}$ has only one nonzero (the pointwise forcing term), $M_{PC} \in R^{n_u \times n_y}$ is supposed to be unknown and each column of $M_{ope} \in R^{n_y \times n_f}$ is the prediction error resulting from the pointwise forcing term represented at the corresponding column of M_{sl}. Noteworthy, for each experiment we take the relation (15) at only one time instant, which depends on the distance between the node and the measurement points; putting more equations in the system from the same experiment gives a final M_{ope} strongly ill-conditioned and we see no advantage, at the moment. Assume we transpose both sides and solve the resulting system, we will have a well-determined system if $n_f = n_y$, an over-determined system if $n_f > n_y$ and an underdetermined system if $n_f < n_y$:

$$M_{ope}^T M_{PC}^T = M_{sl}^T \tag{16}$$

Finally, the feed-forward formula is: recalling expression (14) at the limit $R = \infty$, we want $P_{mf}^T(k) = \bar{P}_{mf}^T(k)$, where the latter is computed from the solution of (16), i.e. $M_{PC}^T = \bar{P}_{mf}^T(k+1)\, C_m^T$. Therefore, we could set

$$Q_{mf}(k)^T C_m^T = \sigma_{Q_{mf}}^2 \; M_{PC}^T - \left(\sigma_{P_f}^2 \; Z + A_m P_{mf}(k) \right)^T C_m^T \tag{17}$$

where $Q_{mf}(k) C_m^T$ is simply a selection of columns of $Q_{mf}(k)$. The choice of the nodes where to apply the pointwise forcing term is crucial and depends on the application. In the model problem here considered, there is an interesting fact: pointwise forcing terms applied to nodes aligned on the same line orthogonal to the measurements boundary, give proportional responses, thus making M_{ope} singular and no unique solution of (16). To avoid it, let us consider to subdivide the nodes in unaligned grids of points, e.g. in Figure 1: let us consider a two-dimensional domain with a regular discretization of points and group them as depicted, i.e. each i-th group of nodes has a different symbol, where only one node of the same group lies on each vertical line; assuming that the bottom horizontal line is the measurements boundary, we have that there aren't two nodes of the same group and aligned orthogonally to that boundary. Then, let us group accordingly the experiments and represent the i-th group of pointwise forcing terms as the columns of $M_{sl,i}$; in this way, we get a set $\{M_{ope,i}\}$ of much

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better conditioned matrices. Then, for each $M_{ope,i}$ build the linear system like (16) and solve it to get the solution $M_{PC,i}$. Then, formula (17) is implemented for each filter of a one-step filter bank, i.e. the feed-forward action $Q_{mf}(k)$ is computed as in (17) for each $M_{PC,i}$ and so the corresponding $\delta\hat{x}(k)$ from (8), that we call $\delta\hat{x}_i(k)$. At the end of this filter-bank computation, the applied $\delta\hat{x}(k)$ is the average of the $\delta\hat{x}_i(k)$ and so $P(k)$ is the average of the updated covariance matrices of each filter in the bank.

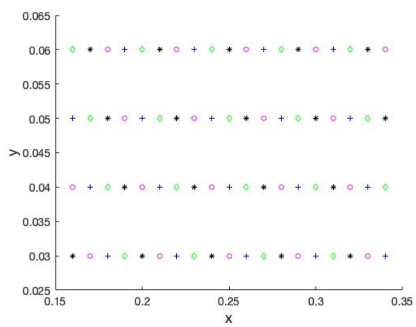

Fig. 1. Unaligned grids of points in the interval $(x, y) \in [0.16, 0.34] \times [0.03, 0.06]$.

Unfortunately, this strategy has some pitfalls and shows in practice a fundamental weakness: to be effective, the covariance matrix of (19), that should be inverted to obtain $P(k)$, becomes very ill-conditioned. However, it suggests an alternative strategy that, as we will, see, gives nice results. Therefore, keeping in mind the linear relation (16), in the next section we see how to extract a reference signal for the feed-forward action and then how to implement this action in addition to the state-update due to the proportional KF gain (13), as usual, indeed, in feedback control theory [12].

3.2 Feed-forward reference extraction

We must obtain a feed-forward reference from the output prediction error, that actually contains also modelling and measurement errors, other than the effect of forcing term estimation error. For this reason, the scheme (15), that uses directly the output prediction error to drive the feed-forward action, is not accurate if entirely used to compute the feed-forward action, as actually done in (17) through the M_{PC} matrix. Here, instead, in order to extract a feed-forward reference signal from the output prediction error, and use it to drive the feed-forward action, we compute the coefficients c_{mq} of a linear combination of M_{ope} columns that approximates the output prediction error, i.e.:

$$M_{ope} c_{mq} = e_{outpred} \tag{18}$$

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We must remember that M_{ope} columns should represent all the nodes where a forcing term should be estimated, and in this way it becomes singular. Note, however, that the columns of M_{ope} are output responses to point-wise loads; hence, each component of c_{mq} is the intensity of a point-wise load located in a mesh node. This suggests an easy regularization scheme on (18) that gives a solution c_{mq} corresponding to a forcing term with some spatial properties, like smoothness, sparsity, etc. In this way the regularization can be tuned accordingly with the specific application, e.g. by exploiting the physical insight. In sec. 3.4 we will propose a general solution. The solution c_{mq}^{reg} of the regularized problem (21) is then multiplied by a gain G_{FF} and becomes the feed-forward contribution to the state estimate $\hat{x}(k-1)$ in (23), which is then used in the usual kalman Filter update (24)-(26). Therefore, the one-step Feed-Forward Augmented Kalman Filter (FF-AKF) algorithm, extended from (6)-(9), becomes:

$$P(k) = \left[\left(Q(k-1) + A(k-1)P(k-1)A(k-1)^T\right)^{-1} + C^T R^{-1} C\right]^{-1} \tag{19}$$

$$e_{outpred} = [C\,(A(k-1)\ \hat{x}(k-1) + B\ u(k-1)) - \bar{y}(k)] \tag{20}$$

$$M_{ope}^{reg}\ c_{mq}^{reg} = e_{outpred} \tag{21}$$

$$\delta\hat{x}_{FF}(k-1) = G_{FF}\ c_{mq}^{reg} \tag{22}$$

$$\hat{x}_{FF}(k) = A(k-1)\ \ (\hat{x}(k-1) + \delta\hat{x}_{FF}(k)) + B\ u(k-1) \tag{23}$$

$$e_{FFoutpred} = [C\hat{x}_{FF}(k) - \bar{y}(k)] \tag{24}$$

$$\delta\hat{x}(k) = -P(k)\ C^T R^{-1} e_{FFoutpred} \tag{25}$$

$$\hat{x}(k) = A(k-1)\ \hat{x}(k-1) + B\ u(k-1) + \delta\hat{x}(k) \tag{26}$$

Note that the feed-forward reference extraction made through the solution of system (18) is blind, i.e. purely algebraic. This means that components of the output prediction error which are due to modelling and/or measurement errors may be confused with pointwise forcing term responses and represented, at least partially, into the coefficients c_{mq}. This would lead to an overestimate of the overall forcing term and create an additional error diffused over the next output prediction errors. For this reason, a safe tuning of the gain G_{FF} is required and seems a practical way to prevent the feed-forward action to create additional noise. This will be confirmed also in the numerical experiments. The tuning of G_{FF} will be explained in the next sec. 3.3 making use of the maximum principle for the heat equation.

3.3 Tuning the Feed-Forward gain G_{FF}

It is easy to see that, according to the well known maximum principle for the heat equation, the effect of an underestimated internal heat source (which is a forcing term in the heat equation) of positive sign is a lower temperature predicted at the boundary with respect to the real, measured, one. Then, to tune conservatively the Feed-Forward gain G_{FF} it may be chosen in such a way

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that the integral of the output prediction error be positive and above a (safety) threshold. For negative heat sources it works the same way, symmetrically.

Moreover, an hysteresis mechanism is conveniently adopted to get a non-zero G_{FF} only if the output prediction error is significantly outside an interval of oscillation where, inside, it may be reasonably dictated by only modeling and measurement errors.

3.4 Regularization issues

We already said that the matrix M_{ope} in (18) is singular and this is due to the fact that it has groups of columns strictly proportional to each other and, important, corresponding to nodes aligned orthogonally to the measured boundary. Therefore, any column in the same group can be picked up in the solution of (18), obtaining the same residual. This means that it will be difficult to estimate accurately the depth of the internal forcing term location, if the regularization is blind. Also, no *column selection* method (sparse recovery or NLLS, see [6], for a recent comparison, and references therein) can give us a good solution without regularization. Then, in this problem, regularization is necessary. Actually, with regularization we then gained a significant improvement from using NNLS, in particular when the heat source is very localized, the regularization parameter should be kept very small and column selection is useful also to solve the required QR factorizations [7], see sec. 4.2.

For this reason, we adopt the subdivision of the nodes in unaligned sets, done in sec. 3.1 (see Figure 1) and consider to solve a system like (18) for each set. It means that each node in the same orthogonal line wrto the border is considered alone in the approximation of the output prediction error, i.e. each depth is considered independently from the others, and then the regularization optimize the shape of the forcing term according to a general criterion.

To do this, we do not solve the (18)-like systems independently, but build a block-diagonal matrix with the $M_{ope,i}$ matrices and regularize it. In this paper we aim at estimating smooth forcing terms, so the regularization is made with a discretization of the laplacian. Actually, we have tried also the TV regularization with quite worse results, as expected for the forcing terms adopted in these experiments, but not true in general. In section 4.2 it will be shown the contribution of this block-diagonal formulation to the correct estimate of the depth of the internal heat source (forcing term). We show also that without regularization the solution concentrates in a greatly overestimated peak, not well centered with the true forcing term, and then wildly oscillates. Indeed, there is a variety of distributed forcing terms that can describe a generic output prediction error, in the problem here considered.

Finally, a consideration must be made about the λ coefficient which weights the regularization term: even knowing the true value of the forcing term, the optimization of λ is not trivial; in particular, optimizing λ at each iteration to minimize the estimation error of the forcing term brings to poor results. This will be future work.

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4 Numerical experiments

In this section, some numerical experiments are described to give a practical evidence of the algorithmic ideas previously presented. In all the following examples, experimental temperatures are simulated numerically, while intensive tests have been done in our previous work [5] to validate the model settings. A note about “inverse crimes": here we are solving the inverse problem using the same model that has generated the data, which is considered an inverse crime. Actually, we are interested in exact reconstructions which are non trivial, and the simplified setting we use is adequate to make significant comparisons, see the interesting discussion in the white-paper of Wirgin [20]. In a real, specific application one should then compare with data containing also model and measurement errors, to validate the practical accuracy of his method in the specific application.

Let us describe the model settings, where the following values of constants are used: $t_f = 1.51\ s$, $L = 0.1\ m$; $\rho C = 3.2 \cdot 10^6\ \frac{J}{m^3\ {}^\circ C}$, $k = 3.77 \cdot 10^3\ \frac{W}{m^\circ C}$. The initial condition is set to $T_0(\cdot) = 20^\circ C$. In this section an Implicit Euler method is adopted for the time discretization, using a temporal step $\Delta t = 0.0005$ in $(0, 0.1]$ and $\Delta t = 0.05$ in $(0.1, t_f]$. A P_1-FE method is used for space discretization, whose step length along y is $h_y = 0.01\ m$, $h_x = h_y$. The sensors are supposed to be in the middle of each mesh edge in the instrumented boundary segment. Numerical experiments have been carried out using Matlab. As a general forcing term we have used a gaussian forcing term f_ϑ with unknown variance and point of application. In the following subsections we see some relevant experiments.

4.1 Inadequacy of the KF proportional action with a diffusive gain

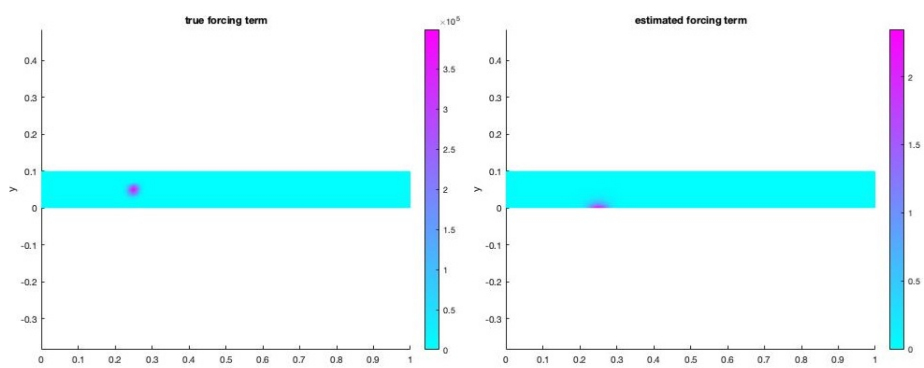

Fig. 2. The true forcing term f_ϑ (left) and its estimate by the standard Augmented Kalman Filter (right) on a rectangular domain $(x, y) \in [0, 1] \times [0, 0.1]$.

The ability of the proposed feed-forward scheme to adequately estimate the intensity and location, and in particular the depth, of localized internal forcing

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terms, is exemplified in Figure 2. We see on the right that, without the contribution of the inverse model of the propagation between pointwise load and output prediction error, described from matrix M_{ope} in sec. 3.1 and used in the FF-AKF algorithm (19)-(26), the estimate of the forcing term is concentrated close to the measured boundary, i.e. in completely wrong position, even if the output prediction error is converging (not shown). On the contrary, in the next section we will show the effectiveness of the feed-forward action, in the same experiment settings.

4.2 The effect of regularization and block-diagonal reference extraction

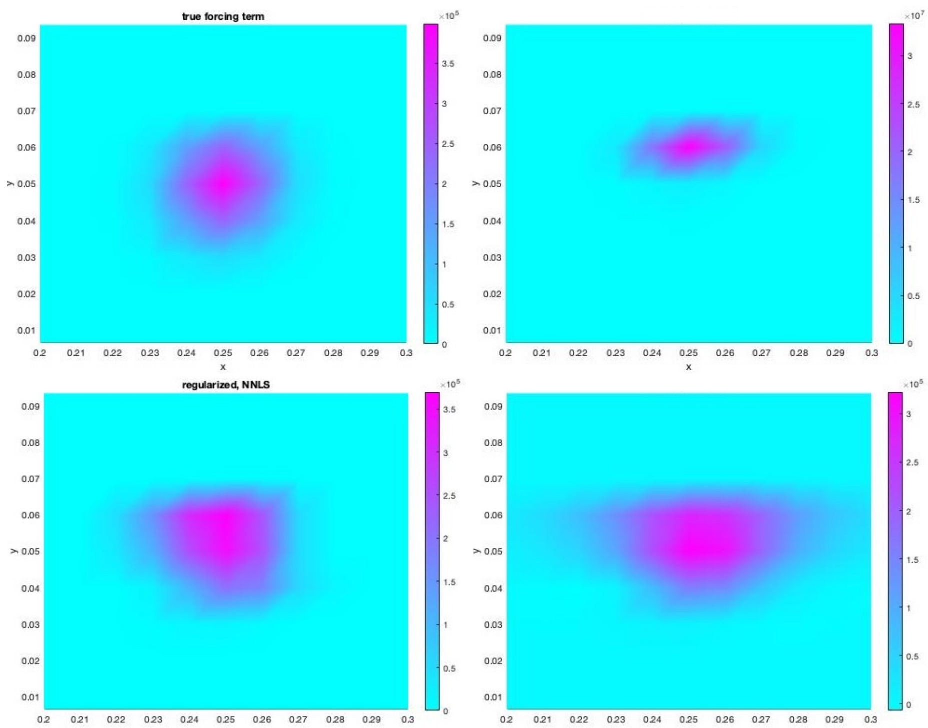

Fig. 3. Zoom in the interval $(x, y) \in [0.2, 0.3] \times [0.01, 0.09]$ of the true heat source f_ϑ (Top-left), its estimate with no regularization (Top-right), the estimate with a regularized NNLS solution (Bottom-left) and the estimate with a regularized OLS solution (Bottom-right).

In Figure 3 we see a zoom of the small portion of the domain where is located the unknown heat source (the whole domain is shown in Figure 2) and the

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corresponding estimates with the FF-AKF algorithm (19)-(26) using/not-using regularization and column selection algorithms. The best result is obtained with regularization and a NonNegative Least Squares (NNLS) solution (Bottom-left). Note that without regularization (Top-right) the magnitude and the support of the estimated forcing term are quite wrong, and with an Ordinary Least Squares solution (Bottom-right) the estimate leaks out from the support of the true forcing term; this is bad also for a correct reconstruction of the temperature field inside the body.

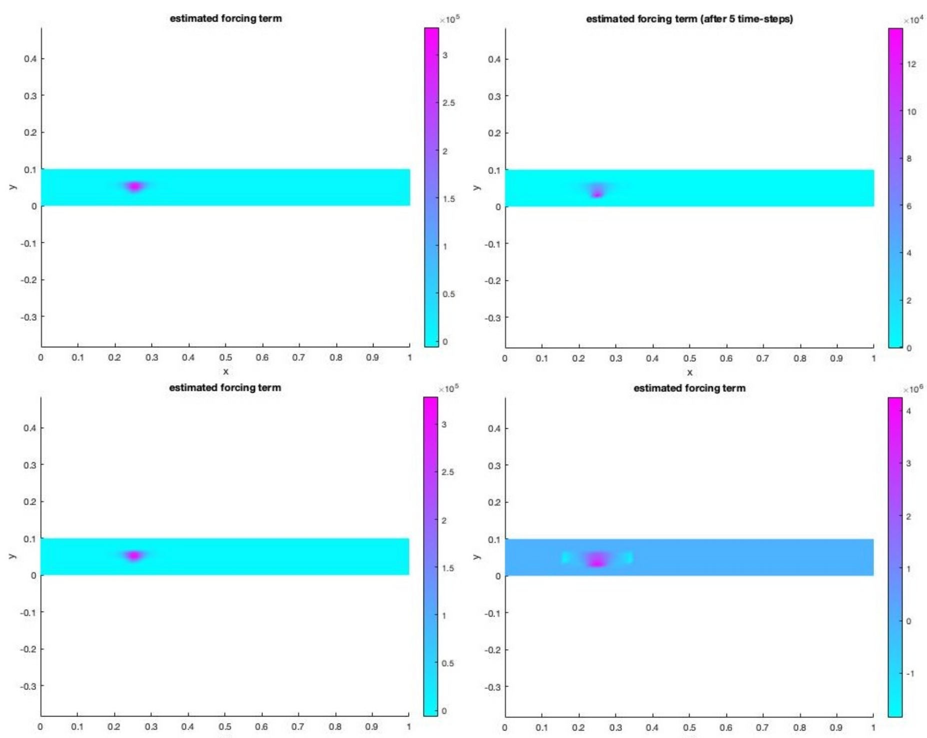

Fig. 4. Top-left: estimate of f_ϑ with a regularized OLS solution with the block-diagonal augmented scheme after a few time-steps (same as Figure 3 Bottom-right); Bottom-left: the same estimate after 40 time-steps; Top/Bottom-right: same as Top/Bottom-left but without the block-diagonal augmented scheme.

In Figure 4 we see the effectiveness of the block-diagonal augmented scheme for feed-forward action determination. On the left, we see the estimated forcing term with a feed-forward action adopting the block-diagonal formulation described in sec. 3.4, after 5 step (Top) and after 40 time-steps (Bottom), showing a good accuracy and stability of the estimate. On the right, we see the estimated forcing term without the block-diagonal formulation after 5 time-steps

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(Top): here it is evident that the nodes closer to the boundary are preferred, thus creating a strong bias on the values and location of the estimated forcing term. On the Bottom-right, we see the estimated forcing term without the block-diagonal formulation after 40 time-steps: here the effect of this bias is amplified by the fact that it has created a too big perturbation in the internal field temperature, that then induces an artifact in the forcing field estimation, in the process of minimizing the output prediction error.

5 Discussion and Conclusions

In this paper we have seen a class of models where many state-variables are not measured and many different combinations of them have quite similar effects on the measured ones. This is often the case with augmented state variables. These *observability* issues [11], cause a non-unique state estimate and the loss of physical interpretability for the computed one. For example, in Figure 2 the standard AKF misses completely the right location and shape of the physical forcing term. The feed-forward technique here proposed, drives the correct estimate through additional information about the physical model. Observability, on the other side, is an algebraic property of the discrete dynamical system and a future direction could be to see if something rigorous can be said about the increase of observability given by this feed-forward technique.

Moreover, this feed-forward technique opens-up a broad range of applications, where the distributed forcing term to be estimated is virtual but equivalent to other kind of unknown perturbations of the system, that are so indirectly estimated, see e.g. [10] [5]; the computational advantage of this analogy is to solve an original nonlinear (e.g. geometric) inverse problem through an equivalent linear inverse problem.
The code of the FF_AKF algorithm is available upon request to the author.

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