Modeling the Dynamics of a Multi-Planetary System with Anisotropic Mass Variation *

Alexander Prokopenya[1[0000−0001−9760−5185]], Mukhtar Minglibayev[2,3[0000−0002−8724−2648]], and Aiken Kosherbayeva[4[0000−0002−8223−2344]]

[1] Warsaw University of Life Sciences–SGGW, Nowoursynowska 159, 02-776, Warsaw, Poland
alexander prokopenya@sggw.edu.pl
[2] Fesenkov Astrophysical Institute, Observatoriya 23, 050020, Almaty, Kazakhstan
minglibayev@gmail.com
[3] Al-Farabi Kazakh National University, Al-Farabi av. 71, 050040, Almaty, Kazakhstan
kosherbaevaayken@gmail.com

Abstract. A classical non-stationary $(n+1)$-body planetary problem with n bodies of variable mass moving around the central star on quasi-elliptic orbits is considered. In addition to the mutual gravitational attraction, the bodies may be acted on by reactive forces arising due to anisotropic variation of their masses. The problem is analyzed in the framework of Newtonian's formalism and the differential equations of motion are derived in terms of the osculating elements of aperiodic motion on quasi-conic sections. These equations can be solved numerically and their solution will describe the motion of the bodies in detail. However, due to the orbital motion of the bodies the perturbing forces include many terms describing short-period oscillations. Therefore, to obtain the solution with high precision one needs to choose very small step size or to use an adoptive step size method and this increase a time of calculation substantially. As we are interested in the long-term behaviour of the system it will be necessary to perform additional calculations in order to extract a secular part of the solution. To simplify the calculations we expand the perturbing forces into power series in terms of eccentricities and inclinations which are assumed to be small and average these equations over the mean longitudes of the bodies. Finally, we obtain the differential equations describing the evolution of orbital parameters over a long period of time. As an application, we have solved the evolution equations numerically in the case of $n = 3$ and demonstrated an influence of the mass variation on the motion of the bodies. All the relevant symbolic and numeric calculations are performed with the aid of the computer algebra system Wolfram Mathematica.

Keywords: Multi-planetary system · variable mass · equations of motion · reactive forces · long-term evolution · Wolfram Mathematica.

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1 Introduction

The classical many-body problem is a famous model of celestial mechanics that is applied for studying an orbital motion in the planetary systems (see [1, 2]). Recall that it describes the dynamical behaviour of the bodies P_0, P_1,... P_n of masses m_0, m_1,... m_n, respectively, attracting each other according to Newton's law of universal gravitation. Such a model provides good approximation for the motion of planets $P_j, (j = 1, 2, \ldots, n)$ around a parent star P_0 if the bodies are spherically symmetric and their masses are constant. Note that applying Newton's second law, one can easily write out the equations of motion of the $(n + 1)$-body system but their general solution cannot be found in the case of three or more interacting bodies.

The mass of the parent star in the planetary system is usually much greater than the masses of planets and so in the first approximation the planets move around the star along Keplerian orbits determined by the corresponding exact solution of the two-body problem. Mutual attraction of the planets disturbs their motion and enforces their orbital parameters to change. However, application of the perturbation theory that has been developed quite well enables to investigate these effects accurately (see [3, 4]). This approach turned out to be very successful for understanding a satellite motion in the Sun-planet and binary star systems when all parameters of the system remain constant and the stationary perturbation theory is used for its analysis (see, for example, [5, 6]).

Real celestial bodies are not always stationary and their characteristics such as mass, size, shape, and internal structure, may vary with time (see, for example, [7–9]). The bodies masses influence essentially on their interaction and motion and so it is natural to study the dynamics of the many-body system with variable masses. Investigation of the simplest such system composed of two bodies has shown that the mass variability affects essentially its dynamic evolution (see [10–14]). Later these investigations were generalized to the system of three bodies of variable masses although works in this field are not numerous (see [15, 16]).

Note that the problem of two bodies of variable masses is not integrable, in general. Therefore, the perturbation theory based on the exact solution of the two-body problem cannot be applied in the case of variable masses. However, one can modify the equations of motion in the problem of two bodies of variable mass in such a way that their general solution can be written in symbolic form for arbitrary law of mass variation of the bodies (see [17]). This solution describes aperiodic motion of a body on a quasi-conic section and may be considered as unperturbed motion. Such approach was exploited in a series of works [18–22], where the problem of three bodies of variable masses was investigated in the framework of the Hamiltonian formalism. Recently the three-body problem was investigated in the framework of Newton's formalism what enables to obtain directly differential equations for the orbital elements (see [23]).

The present work is an extension of [23] and is devoted to the study of dynamical evolution of multi-planetary system of $(n + 1)$ bodies when n planets P_1, P_2,..., P_n move around a central star P_0 on quasi-elliptic orbits which are assumed to not intersect. The problem is studied in the framework of the pertur-

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bation theory where an aperiodic motion on quasi-conic sections is considered as the unperturbed motion. Mutual attraction of the bodies P_1, $P_2, \ldots, P_n$ and reactive forces arising in the case of anisotropic mass variation enforce the orbital elements to change. Differential equations determining the perturbed motion of the bodies are obtained in terms of the osculating elements of aperiodic motion on quasi-conic sections in the framework of Newton's formalism. In the case of small eccentricities and inclinations of the orbits the perturbing forces may be expanded in series in these parameters up to any desired order but here we consider only the first order terms what is sufficient to obtain the results corresponding to the accuracy of the observations. Averaging the equations of the perturbed motion over mean longitudes of the bodies P_1, $P_2, \ldots, P_n$ in the absence of mean-motion resonances, we obtain the differential equations describing the evolution of orbital elements over long periods of time. These equations are solved numerically for different laws of the masses change in the case of $n = 3$. All relevant symbolic and numerical calculations are performed here with the aid of the computer algebra system Wolfram Mathematica [24].

The paper is organized as follows. In Section 2 we describe the model under consideration and obtain the equations of motion in the osculating elements of aperiodic motion on quasi-conic sections. Then in Section 3 derive the evolutionary equations which are solved numerically in Section 4 in the case of $n = 3$. At last, we summarize the results in Conclusion.

2 Equations of motion

In a relative coordinate system with the origin at the center of parent star P_0 of mass $m_0(t)$ the equations of motion of the planets P_1, $P_2, \ldots, P_n$ of masses $m_1(t), m_2(t), ..., m_n(t)$, respectively, may be written in the form (see [17, 23])

$$\frac{d^2\boldsymbol{r}_j}{dt^2} + G(m_0 + m_j)\frac{\boldsymbol{r}_j}{r_j^3} - \frac{\ddot{\gamma}_j}{\gamma_j}\boldsymbol{r}_j = \boldsymbol{F}_j,\ j = 1, 2, \ldots, n. \tag{1}$$

Here G is the gravitational constant, $\boldsymbol{r}_j$ is the radius-vector of the planet P_j and the twice differentiable functions $\gamma_j(t)$ are defined by

$$\gamma_j(t) = \frac{m_{00} + m_{j0}}{m_0(t) + m_j(t)},\ j = 1, 2, \ldots, n, \tag{2}$$

where $m_{00} = m_0(t_0)$, $m_{j0} = m_j(t_0)$ are the masses of the bodies P_0, $P_j, (j = 1, 2, \ldots, n)$, respectively, at the initial instant of time. The forces $\boldsymbol{F}_j$ in the right-hand side of (1) are given by

$$\boldsymbol{F}_j = G\sum_{k=1(k\neq j)}^{n} m_k\left(\frac{\boldsymbol{r}_k - \boldsymbol{r}_j}{r_{jk}^3} - \frac{\boldsymbol{r}_k}{r_k^3}\right) - \frac{\ddot{\gamma}_j}{\gamma_j}\boldsymbol{r}_j + \boldsymbol{Q}_j, \tag{3}$$

where

$$r_{jk} = \sqrt{(x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2},\ r_j = \sqrt{x_j^2 + y_j^2 + z_j^2}, \tag{4}$$

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and the reactive forces $\boldsymbol{Q}_j$ are determined by the expressions (see [25])

$$\boldsymbol{Q}_j = \frac{\dot{m}_j}{m_j}\boldsymbol{V}_j - \frac{\dot{m}_0}{m_0}\boldsymbol{V}_0,\ j = 1, 2, \ldots, n. \quad (5)$$

The dot above a symbol in (3) – (5) denotes the total time derivative of the corresponding function, and $\boldsymbol{V}_j,\ (j = 0, 1, 2, \ldots, n)$ are the relative velocities of the particles leaving the body P_j or falling on it.

2.1 Unperturbed motion

Note that in the case of constant masses when $\gamma_j(t) = 1, (j = 1, 2, \ldots, n)$ equations (1) reduce to the well-known equations determining relative motion of the bodies in the classical $(n+1)$-body problem. These equations are not integrable and are usually studied by methods of perturbation theory using an exact solution of the two-body problem as the first approximation (see, for example, [3]).

To apply similar approach to the case of variable masses we add the terms $\ddot{\gamma}_j/\gamma_j \boldsymbol{r}_j$ in the left-hand side of equations (1) and in expressions (3) for the forces $\boldsymbol{F}_j$ in the right-hand side of (1). This does not change the equations of relative motion (1) but enables to get integrable differential equations from (1) at $\boldsymbol{F}_j = 0$ for arbitrary laws of mass variation of the bodies.

Indeed, at $\boldsymbol{F}_j = 0, (j = 1, 2, \ldots, n)$ equations (1) become independent of each other and each of them has an exact solution that describes aperiodic motion of the body $P_j, (j = 1, 2, \ldots, n)$ on a quasi-conic section (see [17]); it can be written as

$$\begin{aligned} x_j &= \gamma_j a_j \Big((\cos E_j - e_j)(\cos\omega_j \cos\Omega_j - \sin\omega_j \sin\Omega_j \cos i_j) - \\ &\quad - \sqrt{1 - e_j^2} \sin E_j (\sin\omega_j \cos\Omega_j + \cos\omega_j \sin\Omega_j \cos i_j) \Big), \\ y_j &= \gamma_j a_j \Big((\cos E_j - e_j)(\cos\omega_j \sin\Omega_j + \sin\omega_j \cos\Omega_j \cos i_j) - \\ &\quad - \sqrt{1 - e_j^2} \sin E_j (\sin\omega_j \sin\Omega_j - \cos\omega_j \cos\Omega_j \cos i_j) \Big), \\ z_j &= \gamma_j a_j \left((\cos E_j - e_j) \sin\omega_j + \sqrt{1 - e_j^2} \sin E_j \cos\omega_j \right) \sin i_j. \end{aligned} \quad (6)$$

The constants a_j, e_j, i_j, Ω_j and ω_j in (6) are analogues of the well-known Keplerian orbital elements and are determined from the initial conditions of motion (see [17]). An analogue of the eccentric anomaly E_j is determined by the well-known Kepler equation

$$E_j - e_j \sin E_j = M_j, \quad (7)$$

where analog of the mean anomaly M_j is given by

$$M_j = \frac{\sqrt{\kappa_j}}{a_j^{3/2}} \left(\Phi_j(t) - \Phi_j(\tau_j) \right), \quad \Phi_j(t) = \int_0^t \frac{dt}{\gamma_j^2(t)}\ , \quad (8)$$

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where $\kappa_j = G(m_{00} + m_{j0}), (j = 1, 2, \ldots, n)$. By τ_j in (8) we denote an analog of the time when the body P_j passes through the pericenter.

Note that solutions (6) differ from the corresponding solutions to the two-body problem with constant masses only by the presence of a time-dependent scaling coefficient $\gamma_j(t)$. Besides, the mean anomaly M_j is not a linear function of time but it is an increasing function of time (see (8)). If the laws of masses variation $m_j(t)$ are known the functions $\gamma_j(t)$ define the mean anomalies $M_j(t)$ and equation (7) enables to find the eccentric anomalies $E_j(t)$ as functions of time. Therefore, solutions (6) define the unperturbed motion of the planets P_j in terms of the time and $6n$ constants of integration $a_j, e_j, i_j, \Omega_j, \omega_j$, and $\tau_j, j = 1, 2 \ldots, n$, which may be considered as analogues of the Keplerian orbital elements (see [17]).

2.2 Perturbed motion

Mutual attraction and reactive forces (5) arising in the case of anisotropic mass variation of the bodies P_j affect their motion and the orbital elements must necessarily vary with the time. To obtain the differential equations, determining the dependence of the orbital parameters on time, one can use the method of the variation of arbitrary constants that is well-known in the theory of differential equations. Assuming the orbital parameters are functions of time and substituting solutions (6) into (1), we obtain $3n$ differential equations for $6n$ unknown functions $a_j(t)$, $e_j(t)$, $i_j(t)$, $\Omega_j(t)$, $\omega_j(t)$, $M_j(t), (j = 1, 2, \ldots, n)$. Additional $3n$ equations are usually obtained from the condition that the coordinates x_j, y_j, z_j and the corresponding velocity components at time t are determined by functions (6) and their derivatives with respect to time under the condition that the orbital elements are constant. As a result, the perturbed coordinates and velocity components of the bodies P_j yield the instantaneous orbital elements $a_j, e_j, i_j, \Omega_j, \omega_j$, and M_j given by formulas (6)–(8). Such instantaneous elements are known as the osculating elements (see, for example, [1, 2]).

By performing the corresponding symbolic calculations (see details in [23]), we obtain the following system of differential equations for finding the dependence of the orbital elements on time:

$$\frac{da_j}{dt} = \frac{2a_j^{3/2}\gamma_j(t)}{\sqrt{\kappa_j}(1 - e_j \cos E_j)} \left(e_j \sin E_j \ F_{rj} + \sqrt{1 - e_j^2} F_{\tau j} \right), \tag{9}$$

$$\frac{de_j}{dt} = \frac{\sqrt{a_j(1 - e_j^2)}\gamma_j(t)}{\sqrt{\kappa_j}(1 - e_j \cos E_j)} \left(\sqrt{1 - e_j^2} \sin E_j \ F_{rj} + \right.$$
$$\left. + \left(2 \cos E_j - e_j - e_j \cos^2 E_j\right) F_{\tau j} \right), \tag{10}$$

$$\frac{di_j}{dt} = \frac{\sqrt{a_j}\gamma_j(t)}{\sqrt{\kappa_j(1 - e_j^2)}} F_{nj} \left((\cos E_j - e_j) \cos \omega_j - \sqrt{1 - e_j^2} \sin \omega_j \sin E_j \right), \tag{11}$$

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$$\frac{d\Omega_j}{dt} = \frac{\sqrt{a_j}\gamma_j(t)}{\sqrt{\kappa_j(1-e_j^2)}}\ \frac{F_{nj}}{\sin i_j}\left((\cos E_j - e_j)\sin\omega_j + \sqrt{1-e_j^2}\cos\omega_j\sin E_j\right), \quad (12)$$

$$\frac{d\omega_j}{dt} = -\frac{\sqrt{a_j}\gamma_j(t)\cot i_j}{\sqrt{\kappa_j(1-e_j^2)}}F_{nj}\left((\cos E_j - e_j)\sin\omega_j + \sqrt{1-e_j^2}\cos\omega_j\sin E_j\right) -$$
$$-\frac{\sqrt{a_j}\gamma_j(t)}{e_j\sqrt{\kappa_j}(1-e_j\cos E_j)}\left((\cos E_j - e_j)\sqrt{1-e_j^2}F_{rj} - \right.$$
$$\left. -(2-e_j^2-e_j\cos E_j)\sin E_j F_{\tau j}\right), \quad (13)$$

$$\frac{dM_j}{dt} = \frac{\sqrt{a_j}\gamma_j(t)}{e_j\sqrt{\kappa_j}(1-e_j\cos E_j)}\left(\sqrt{1-e_j^2}(-2+e_j^2+e_j\cos E_j)\sin E_j F_{\tau j} + \right.$$
$$\left. +\left((1+3e_j^2)\cos E_j - e_j(3+e_j^2\cos(2E_j))\right)F_{rj}\right) + \frac{\sqrt{\kappa_j}}{a_j^{3/2}\gamma_j^2(t)}. \quad (14)$$

The forces F_{rj}, $F_{\tau j}$, and F_{nj} in the right-hand sides of (9)–(14) are the radial, transversal and normal components of the forces $\boldsymbol{F}_j$, respectively, determined by expressions (3), (5). The reactive forces $\boldsymbol{Q}_j$ (see (5)) are usually determined in the orbital systems of coordinates of the bodies P_j, so the forces $\boldsymbol{F}_j$ are also written in these systems of coordinates. The direction cosines of the unit vectors $\boldsymbol{e}_{rj} = (e_{xj}, e_{yj}, e_{zj})$, $\boldsymbol{e}_{\tau j} = (\tau_{xj}, \tau_{yj}, \tau_{zj})$, and $\boldsymbol{e}_{nj} = (n_{xj}, n_{yj}, n_{zj})$ along the radial, transversal, and normal directions, respectively, can be easily written on the basis of solutions (6):

$$e_{xj} = \frac{x_j}{\gamma_j a_j},\ e_{yj} = \frac{y_j}{\gamma_j a_j},\ e_{zj} = \frac{z_j}{\gamma_j a_j}, \quad (15)$$

$$n_{xj} = \sin\Omega_j\sin i_j,\ n_{yj} = -\cos\Omega_j\sin i_j,\ n_{zj} = \cos i_j, \quad (16)$$

$$\tau_{xj} = n_{yj}e_{zj} - n_{zj}e_{yj},\ \tau_{yj} = n_{zj}e_{xj} - n_{xj}e_{zj},\ \tau_{zj} = n_{xj}e_{yj} - n_{yj}e_{xj}. \quad (17)$$

Denoting the components of the relative velocities of particles leaving the body $P_j, (j = 1, \ldots, n)$ or falling on them along the radial, transversal, and normal directions in the orbital system of coordinates related to the body P_j by V_{rj}, $V_{\tau j}$, V_{nj} and using (3), (5), we obtain

$$F_{rj} = \boldsymbol{F}_j\cdot\boldsymbol{e}_{rj} = G\sum_{k=1(k\neq j)}^{n} m_k\left(\left(\frac{r_k}{r_{jk}^3} - \frac{1}{r_k^2}\right)(\boldsymbol{e}_{rk}\cdot\boldsymbol{e}_{rj}) - \frac{r_j}{r_{jk}^3}\right) - \frac{\ddot{\gamma}_j}{\gamma_j}r_j + Q_{rj},$$

$$F_{\tau j} = \boldsymbol{F}_j\cdot\boldsymbol{e}_{\tau j} = G\sum_{k=1(k\neq j)}^{n} m_k\left(\frac{r_k}{r_{jk}^3} - \frac{1}{r_k^2}\right)(\boldsymbol{e}_{rk}\cdot\boldsymbol{e}_{\tau j}) + Q_{\tau j},$$

$$F_{nj} = \boldsymbol{F}_j\cdot\boldsymbol{e}_{nj} = G\sum_{k=1(k\neq j)}^{n} m_k\left(\frac{r_k}{r_{jk}^3} - \frac{1}{r_k^2}\right)(\boldsymbol{e}_{rk}\cdot\boldsymbol{e}_{nj}) + Q_{nj}, \quad (18)$$

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where the corresponding components of the reactive forces $\boldsymbol{Q}_j$ are given by

$$Q_{rj} = \frac{\dot{m}_j}{m_j} V_{rj} - \frac{\dot{m}_0}{m_0} \left(V_{r0} \left(\boldsymbol{e}_{r1} \cdot \boldsymbol{e}_{rj} \right) + V_{\tau 0} \left(\boldsymbol{e}_{\tau 1} \cdot \boldsymbol{e}_{rj} \right) + V_{n0} \left(\boldsymbol{e}_{n1} \cdot \boldsymbol{e}_{rj} \right) \right),$$

$$Q_{\tau j} = \frac{\dot{m}_j}{m_j} V_{\tau j} - \frac{\dot{m}_0}{m_0} \left(V_{r0} \left(\boldsymbol{e}_{r1} \cdot \boldsymbol{e}_{\tau j} \right) + V_{\tau 0} \left(\boldsymbol{e}_{\tau 1} \cdot \boldsymbol{e}_{\tau j} \right) + V_{n0} \left(\boldsymbol{e}_{n1} \cdot \boldsymbol{e}_{\tau j} \right) \right), \quad (19)$$

$$Q_{nj} = \frac{\dot{m}_j}{m_j} V_{nj} - \frac{\dot{m}_0}{m_0} \left(V_{r0} \left(\boldsymbol{e}_{r1} \cdot \boldsymbol{e}_{nj} \right) + V_{\tau 0} \left(\boldsymbol{e}_{\tau 1} \cdot \boldsymbol{e}_{nj} \right) + V_{n0} \left(\boldsymbol{e}_{n1} \cdot \boldsymbol{e}_{nj} \right) \right).$$

The relative velocities $\boldsymbol{V}_0$ in (19) of the particles leaving the body P_0 or falling on it are assumed to be given in the orbital system of coordinates related to the body P_1. If the relative velocities $\boldsymbol{V}_0$ and $\boldsymbol{V}_j$ and laws of variation of body masses are known, equations (9) – (14) completely determine the perturbed motion of the bodies $P_j, j = 1, 2, \ldots, n$.

3 Evolutionary Equations

Differential equations (9) – (14) describe the perturbed motion of the planets in terms of the osculating orbital elements but they are not integrable and their exact solution cannot be found. However, in many problems of celestial mechanics, eccentricities and inclinations of body orbits are small (see [1, 4]). Here we consider this practically important case of small eccentricities $e_j << 1$ and inclinations $i_j << 1, (j = 1, 2, \ldots, n)$ and expand the right-hand sides of equations (9) – (14) in power series in these parameters. Note that applying the computer algebra system *Mathematica* (see [24]), one can calculate such expansions with any required accuracy but the corresponding expressions become very cumbersome in higher order terms. Here we restrict ourselves to computations up to the first order and obtain the following differential equations for the secular perturbations of the orbital elements of the body P_1:

$$\frac{da_1}{dt} = \frac{2a_1^{3/2}\gamma_1}{\sqrt{\kappa_1}} \left(\frac{\dot{m}_1}{m_1} V_{\tau 1} - \frac{\dot{m}_0}{m_0} V_{\tau 0} \right),$$

$$\frac{de_1}{dt} = -\frac{3\sqrt{a_1}}{2\sqrt{\kappa_1}} e_1 \gamma_1 \left(\frac{\dot{m}_1}{m_1} V_{\tau 1} - \frac{\dot{m}_0}{m_0} V_{\tau 0} \right) + \sum_{s=2}^{n} \frac{G m_s e_s}{\sqrt{a_1 \kappa_1}} \Pi_{12}^{1s} \sin(\omega_1 - \omega_s + \Omega_1 - \Omega_s),$$

$$\frac{di_1}{dt} = -\frac{3\sqrt{a_1}}{2\sqrt{\kappa_1}} e_1 \gamma_1 \left(\frac{\dot{m}_1}{m_1} V_{n1} - \frac{\dot{m}_0}{m_0} V_{n0} \right) \cos\omega_1 + \sum_{s=2}^{n} \frac{G m_s i_s}{4\sqrt{a_1 \kappa_1}} B_1(\alpha_{1s}) \sin(\Omega_1 - \Omega_s),$$

$$\frac{d\Omega_1}{dt} = -\frac{3\sqrt{a_1}}{2\sqrt{\kappa_1}} e_1 \gamma_1 \left(\frac{\dot{m}_1}{m_1} V_{n1} - \frac{\dot{m}_0}{m_0} V_{n0} \right) \frac{\sin\omega_1}{i_1} -$$

$$- \sum_{s=2}^{n} \frac{G m_s}{4\sqrt{a_1 \kappa_1}} B_1(\alpha_{1s}) \left(1 - \frac{i_s}{i_1} \cos(\Omega_1 - \Omega_s) \right),$$

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$$\frac{d\omega_1}{dt} = \frac{3\sqrt{a_1}}{2\sqrt{\kappa_1}} e_1\gamma_1 \left(\frac{\dot{m}_1}{m_1}V_{n1} - \frac{\dot{m}_0}{m_0}V_{n0}\right)\frac{\sin\omega_1}{i_1} + \frac{\sqrt{a_1}}{\sqrt{\kappa_1}}\gamma_1\left(\frac{\dot{m}_1}{m_1}V_{r1} - \frac{\dot{m}_0}{m_0}V_{r0}\right) -$$

$$-\frac{3a_1^{3/2}}{2\sqrt{\kappa_1}}\gamma_1\ddot{\gamma}_1 + \sum_{s=2}^{n}\frac{Gm_s}{\sqrt{a_1\kappa_1}}\left(\varPi_{11}^{1s} - \frac{1}{4}B_1(\alpha_{1s})\left(1 + \frac{i_s}{i_1}\cos(\varOmega_1 - \varOmega_s)\right)\right) +$$

$$+\sum_{s=2}^{n}\frac{Gm_s}{\sqrt{a_1\kappa_1}}\frac{e_s}{e_1}\varPi_{12}^{1s}\cos(\omega_1 - \omega_s + \varOmega_1 - \varOmega_s),\ \alpha_{1s} = \frac{a_1\gamma_1}{a_s\gamma_s} < 1. \tag{20}$$

Remind that reactive forces (19) acting on the star P_0 are determined in the orbital coordinate system of the body P_1. Due to this equations (20) differ a little bit of the differential equations for the secular perturbations of the orbital elements of the bodies $P_2, \ldots, P_n$ which are given by

$$\frac{da_j}{dt} = \frac{2a_j^{3/2}\gamma_j}{\sqrt{\kappa_j}}\frac{\dot{m}_j}{m_j}V_{\tau j},$$

$$\frac{de_j}{dt} = -\frac{3\sqrt{a_j}}{2\sqrt{\kappa_j}}e_j\gamma_j\frac{\dot{m}_j}{m_j}V_{\tau j} + \frac{3\sqrt{a_j}}{2\sqrt{\kappa_j}}e_j\gamma_j\frac{\dot{m}_0}{m_0}\left(e_1V_{\tau 0}\cos(\omega_1 - \omega_j + \varOmega_1 - \varOmega_j\right) +$$

$$+ e_1V_{r0}\sin(\omega_1 - \omega_j + \varOmega_1 - \varOmega_j) + i_1V_{n0}\cos(\omega_j - \varOmega_1 + \varOmega_j) - i_jV_{n0}\cos\omega_j) -$$

$$-\sum_{s=1}^{j-1}\frac{Gm_se_s}{\sqrt{a_j\kappa_j}}\varPi_{12}^{sj}\sin(\omega_s - \omega_j + \varOmega_s - \varOmega_j) +$$

$$+\sum_{s=j+1}^{n}\frac{Gm_se_s}{\sqrt{a_j\kappa_j}}\varPi_{12}^{js}\sin(\omega_j - \omega_s + \varOmega_j - \varOmega_s),$$

$$\frac{di_j}{dt} = -\frac{3\sqrt{a_j}}{2\sqrt{\kappa_j}}e_j\gamma_j\left(\frac{\dot{m}_j}{m_j}V_{nj} - \frac{\dot{m}_0}{m_0}V_{n0}\right)\cos\omega_j -$$

$$-\sum_{s=1}^{j-1}\frac{Gm_si_s}{4\sqrt{a_j\kappa_j}}B_1(\alpha_{sj})\sin(\varOmega_s - \varOmega_j) + \sum_{s=j+1}^{n}\frac{Gm_si_s}{4\sqrt{a_j\kappa_j}}B_1(\alpha_{js})\sin(\varOmega_j - \varOmega_s),$$

$$\frac{d\varOmega_j}{dt} = -\frac{3\sqrt{a_j}}{2\sqrt{\kappa_j}}e_j\gamma_j\left(\frac{\dot{m}_j}{m_j}V_{nj} - \frac{\dot{m}_0}{m_0}V_{n0}\right)\frac{\sin\omega_j}{i_j} -$$

$$-\sum_{s=1}^{j-1}\frac{Gm_s}{4\sqrt{a_j\kappa_j}}B_1(\alpha_{sj})\left(1 - \frac{i_s}{i_j}\cos(\varOmega_s - \varOmega_j)\right) -$$

$$-\sum_{s=j+1}^{n}\frac{Gm_s}{4\sqrt{a_j\kappa_j}}B_1(\alpha_{js})\left(1 - \frac{i_s}{i_j}\cos(\varOmega_j - \varOmega_s)\right),$$

$$\frac{d\omega_j}{dt} = -\frac{3a_j^{3/2}}{2\sqrt{\kappa_j}}\gamma_j\ddot{\gamma}_j + \frac{3\sqrt{a_j}}{2\sqrt{\kappa_j}}e_j\gamma_j\left(\frac{\dot{m}_j}{m_j}V_{nj} - \frac{\dot{m}_0}{m_0}V_{n0}\right)\frac{\sin\omega_j}{i_j} +$$

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$$+\frac{\sqrt{a_j}}{\sqrt{\kappa_j}}\gamma_j\frac{\dot{m}_j}{m_j}V_{rj}+\frac{3\sqrt{a_j}}{2e_j\sqrt{\kappa_j}}\gamma_j\frac{\dot{m}_0}{m_0}\left(V_{n0}(i_j\sin\omega_j-i_1\sin(\omega_j-\Omega_1+\Omega_j))-\right.$$

$$\left.-V_{r0}e_1\cos(\omega_1-\omega_j+\Omega_1-\Omega_j)+V_{\tau 0}e_1\sin(\omega_1-\omega_j+\Omega_1-\Omega_j)\right)+$$

$$+\sum_{s=1}^{j-1}\frac{Gm_s}{\sqrt{a_j\kappa_j}}\left(\Pi_{22}^{sj}-\frac{1}{4}B_1(\alpha_{sj})\left(1+\frac{i_s}{i_j}\cos(\Omega_s-\Omega_j)\right)+\right.$$

$$\left.+\frac{e_s}{e_j}\Pi_{12}^{sj}\cos(\omega_s-\omega_j+\Omega_s-\Omega_j)\right)+$$

$$+\sum_{s=j+1}^{n}\frac{Gm_s}{\sqrt{a_j\kappa_j}}\left(\Pi_{11}^{js}-\frac{1}{4}B_1(\alpha_{js})\left(1+\frac{i_s}{i_j}\cos(\Omega_j-\Omega_s)\right)+\right.$$

$$\left.+\frac{e_s}{e_j}\Pi_{12}^{js}\cos(\omega_j-\omega_s+\Omega_j-\Omega_s)\right),\ j=2,3,\ldots,n. \tag{21}$$

Here

$$\Pi_{12}^{ik}=\frac{1}{8}\left(9B_0(\alpha_{ik})+B_2(\alpha_{ik})\right)-\frac{1+\alpha_{ik}^2}{8\alpha_{ik}}\left(9C_0(\alpha_{ik})-3C_2(\alpha_{ik})\right)+$$

$$+\frac{3}{16}\left(7C_1(\alpha_{ik})+C_3(\alpha_{ik})\right),$$

$$\Pi_{11}^{ik}=-\frac{3}{4}\alpha_{ik}\left(B_0(\alpha_{ik})+2C_1(\alpha_{ik})\right)+\frac{6\alpha_{ik}^2+15}{8}C_0(\alpha_{ik})-\frac{9}{8}C_2(\alpha_{ik}),$$

$$\Pi_{22}^{ik}=-\frac{3}{4\alpha_{ik}}\left(B_0(\alpha_{ik})+2C_1(\alpha_{ik})\right)+\frac{6+15\alpha_{ik}^2}{8\alpha_{ik}^2}C_0(\alpha_{ik})-\frac{9}{8}C_2(\alpha_{ik}), \tag{22}$$

and $B_0(\alpha_{ik}), B_1(\alpha_{ik}), B_2(\alpha_{ik}), C_0(\alpha_{ik}), C_1(\alpha_{ik}), C_2(\alpha_{ik}), C_3(\alpha_{ik})$ are the Laplace coefficients (see [4, 23]). As orbital parameters of the bodies are assumed to satisfy the conditions $a_1\gamma_1 < a_2\gamma_2 < \ldots < a_n\gamma_n$ the arguments of the Laplace coefficients in (20)– (22) are smaller than 1:

$$\alpha_{ik}=\frac{a_i\gamma_i}{a_k\gamma_k}<1,\quad 1\le i<k\le n. \tag{23}$$

Equations (20), (21) determine the secular perturbations of the orbital elements of the planets $P_1,\ldots,P_n$. Although we take into account only linear terms in the power expansions of the right-hand sides of equations (9) – (14) in terms of eccentricities e_j and inclinations i_j, the equations (20), (21) are very complicated and we cannot find their solution in symbolic form. However, we can choose some realistic laws of the masses variations and find their numerical solution. In this way we can investigate an influence of the masses variation on the dynamics of the $(n+1)$-body planetary system.

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4 Simulation

To test the model, let us consider the case of three planets P_1, P_2, P_3 orbiting the parent star P_0. To solve equations (20), (21) numerically, it is expedient to use the dimensionless variables. For example, we use initial values of the semi-major axis $a_{10} = a_1(t_0)$ and the mass m_{00} of body P_0 as units of distance and mass, respectively, and define dimensionless distance a_j^*, mass m_j^* and time t^* by

$$a_j^* = \frac{a_j}{a_{10}},\ m_0^* = \frac{m_0}{m_{00}},\ m_j^* = \frac{m_j}{m_{00}},\ t^* = t\frac{\sqrt{\kappa_1}}{a_{10}^{3/2}},\ j = 1, 2, 3. \tag{24}$$

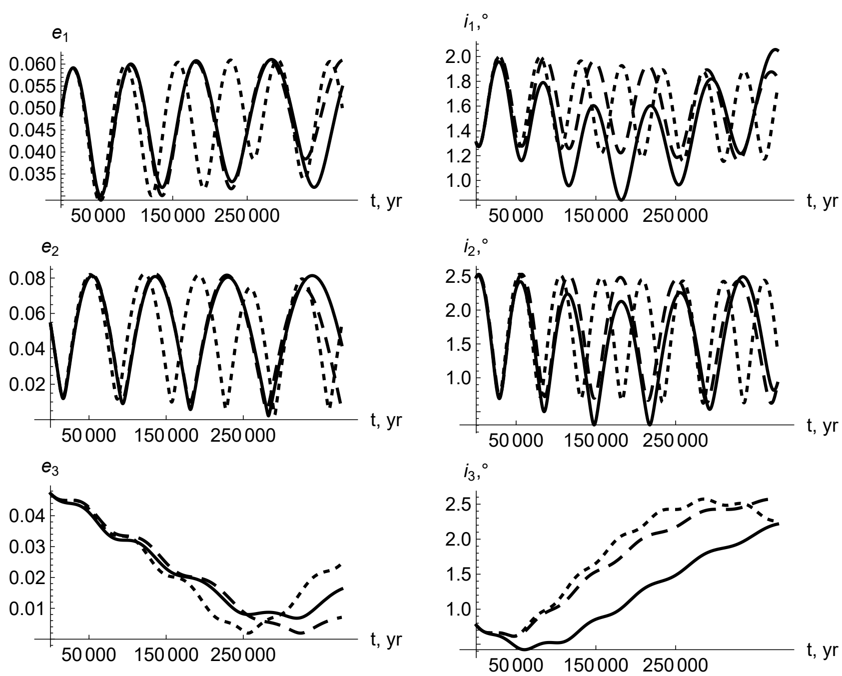

Fig. 1. Eccentricities e_j and inclinations i_j of the bodies P_1, P_2, P_3 (short dashing – constant masses, long dashing – isotropic mass changes, solid curves – anisotropic mass changes, $V_{n0} = 1/2, V_{\tau 1} = -1$).

The mass of the parent star P_0 is assumed to decrease according to the Eddington-Jeans law

$$m_0^*(t^*) = \left((m_{00}^*)^{1-n_j} - \beta_0(1-n_0)(t^* - t_0^*)\right)^{\frac{1}{1-n_0}}, \tag{25}$$

where $n_0 = 2,\ \beta_0 = 1/300000$, while the mass of the planet P_1 increases with time at constant dimensionless rate $\dot{m}_1 = 2,277 \cdot 10^{-12}$. Masses of the planets P_2, P_3 are assumed to be constant.

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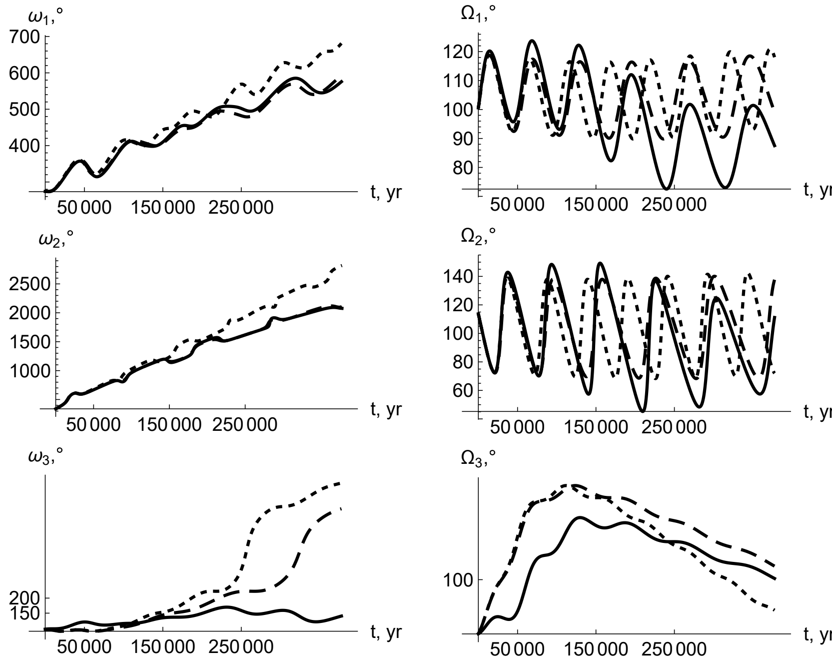

Fig. 2. Parameters ω_j and Ω_j of the bodies P_1, P_2, P_3 (short dash – constant masses, long dash – isotropic mass changes, solid curves – anisotropic mass changes, $V_{n0} = 1/2$, $V_{\tau 1} = -1$).

As a test system, we consider the Sun, Jupiter, Saturn, and Uranus as bodies P_0, P_1, P_2, and P_3, respectively, and choose the following initial values for orbital elements (see [4]):

$$m_{00} = 1,9891 \times 10^{30} kg,\ m_{10} = 1,8982 \times 10^{27} kg,\ m_{20} = 5,6852 \times 10^{26} kg,$$
$$m_{30} = 8,6843 \times 10^{25} kg,\ a_{10} = 5,2034 AU,\ a_{20} = 9,5371 AU,\ a_{30} = 19,191 AU,$$
$$e_{10} = 0,0484,\ e_{20} = 0,0541,\ e_{30} = 0,0472,\ i_{10} = 1,304^\circ,\ i_{20} = 2,485^\circ,$$
$$i_{30} = 0,772^\circ,\ \Omega_{10} = 100,56^\circ,\ \Omega_{20} = 113,72^\circ,\ \Omega_{30} = 74,23^\circ,$$
$$\omega_{10} = 273,98^\circ,\ \omega_{20} = 338,71^\circ,\ \omega_{30} = 96,73^\circ.$$

In the case of constant masses of the bodies equations (20), (21) describe the secular perturbations of the orbital elements in the framework of the classical four-body problem (see [4], [2]). Taking into account the isotropic mass variation of the body P_0 according to the Eddington-Jeans law and linear isotropic increase of mass of the body P_1 when reactive forces do not arise results in only some quantitative changes of solutions to (20), (21) (see Fig. 1,2). However, the anisotropic mass variation with only two nonzero dimensionless velocities $V_{n0} = 1/2, V_{\tau 1} = -1$ modifies substantially behaviour of the inclinations i_j and the longitudes of the ascending node Ω_j of all three planets. The semi-major

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axes a_2 and a_3 remain constant because only transversal reactive forces $Q_{\tau j}$ can change them (see (20), (21)). As we assume $V_{\tau 1} = -1$ the semi-major axis a_1 of the body P_1 decreases with time (Fig. 3).

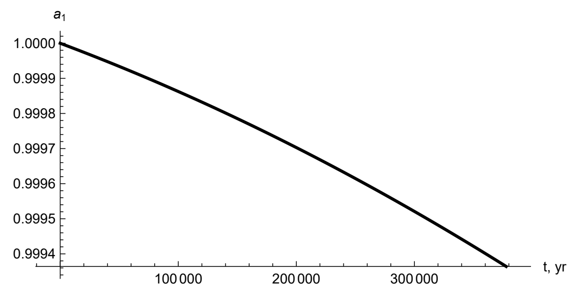

Fig. 3. Semi-major axis a_1 of the body P_1 in the case of $V_{n0} = 1/2, V_{\tau 1} = -1$).

If only one component of the relative velocity V_{r0} of the particles leaving the most massive body P_0 along the radial direction is greater than zero ($V_{r0} = 1$) dependance of the eccentricities e_j and arguments of pericenter ω_j on time changes (see Fig. 4,5). Again orbital elements of the body P_3 are the most sensitive to appearance of the radial reactive force because its mass is the smallest one. These results demonstrate that even very small changes of the masses of celestial bodies can influence essentially on their long-term evolution.

5 Conclusion

In this paper, we investigate a multi-planetary problem of many bodies of variable masses that attract each other according to Newton's law of universal gravitation. We assume that the bodies may be acted on by the reactive forces arising due to anisotropic variation of the bodies masses. Using Meshcherskii equation, we have defined the reactive forces explicitly and derived the differential equations of motion of the bodies in the relative system of coordinates with the most massive body P_0 located at the origin in the framework of Newton's formalism. Equations of motion (1) are presented in the form which enables to find an exact solution (6) to the two-body problem of variable masses in the case of $\boldsymbol{F}_j = 0$. Using the exact solution (6) and applying the method of variation of constants, we derived differential equations of the perturbed motion in terms of the osculating elements of the aperiodic motion on quasi-conical section. It should be emphasized that the obtained equations (9)–(14) are valid for any laws of the mass variation of the bodies and completely determine the perturbed motion of the bodies P_j, $(j = 1, 2, \ldots, n)$.

In the case of small eccentricities and inclinations of orbits, we have expanded the right-hand sides of equations (9)–(14) in power series in terms of the orbital elements up to the first order. As the coefficients of e_j and i_j in the obtained

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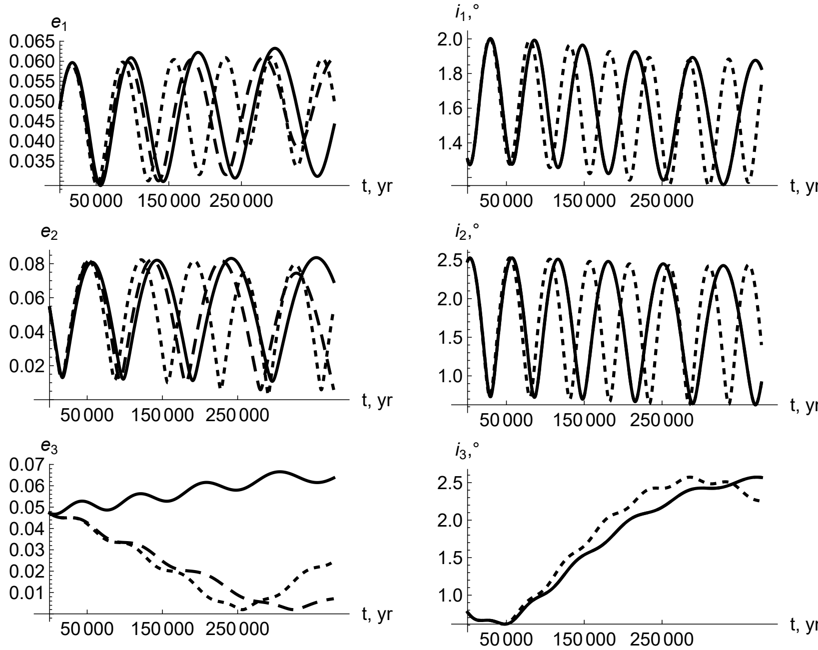

Fig. 4. Eccentricities e_j and inclinations i_j of the bodies P_1, P_2, P_3 (short dash – constant masses, long dash – isotropic mass changes, solid curves – anisotropic mass changes, $V_{r0} = 1$).

expressions are periodic functions of the mean longitudes λ_j, we replaced them by the corresponding Fourier series. Finally, we have shown that the right-hand sides of differential equations (9)–(14) contain the terms describing behaviour of the orbital elements on long-time intervals and quite cumbersome terms determining the short-term oscillations of the orbital elements. Assuming that the mean-motion resonances are absent in the system and averaging the equations over the mean longitudes λ_j, we derived the differential equations determining the secular perturbations of the orbital elements. Note that the equations obtained describe the perturbed motion of the bodies in the general case when the masses of all bodies vary anisotropically, and reactive forces occur.

To test the model, we have solved the averaged equations (20), (21) numerically for three planets ($n = 3$) in the case when the mass of the parent star decreases according to the Eddington-Jeans law and the mass of the body P_1 increases linearly with time while masses of the bodies P_2, P_3 do not change. The results obtained in two different cases of reactive forces acting on the bodies P_0, P_1 are shown on Fig. 1–5. Comparison with the case of constant masses (see, for example, [4]) demonstrates that masses variation can significantly affect the evolution of orbital parameters. Thus, choosing some realistic values of the system parameters and different laws of the mass variation one can solve equations (20), (21) numerically and investigate an influence of the masses variation on

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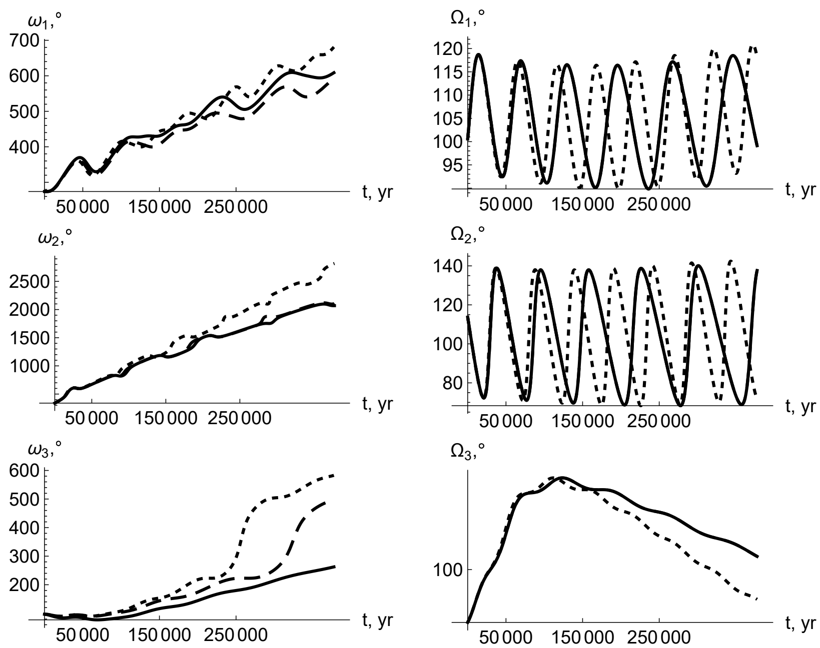

Fig. 5. Parameters ω_j and Ω_j of the bodies P_1, P_2, P_3 (short dash – constant masses, long dash – isotropic mass changes, solid curves – anisotropic mass changes, $V_{r0} = 1$).

the dynamics of multi-planetary systems. In our future work, we plan to study exoplanetary systems of variable masses when the number of celestial bodies exceeds four.

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