Computational aspects of homogeneous approximations of nonlinear systems

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Abstract. The objective of the paper is to describe computational methods and techniques of investigation of certain algebraic structures needed in order to apply the results in concrete problems in mathematical control theory of nonlinear systems. Contemporary theoretic research requires more and more sophisticated tools for a possible application of the results. In the paper we propose computational tools and techniques for a certain type of simplification of driftless control systems. Such simplification still preserve most crucial properties of the original ones like controllability, but the simplified system have a special feedforward form that is much easier to integrate or allows to solve other problems in control theory. We present the computational procedure and foundations of the library as the extension of existing software libraries in Python language. The approach is illustrated with some numerical experiments and simulations. We conclude with a discussion about related computational issues.

Keywords: nonlinear system · nonlinear approximation · homogeneous approximation · computational procedures.

1 Introduction

We consider control systems – nonlinear with respect to state and linear with respect to controls – namely (driftless) systems of the form

$$\dot{x} = \sum_{i=1}^{m} X_i(x) u_i, \tag{1}$$

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where $X_i(x)$ are real analytic vector fields in the neighborhood of the origin in $\mathbb{R}^n$. It is one of class of systems widely considered in modern control theory. In general, control theory is devoted to studying dynamical processes that are described in most cases as systems of difference or differential equations (usually with additional constraints) where some parameters – controls – can be externally changed at any moment of time during the process, as desired. One of most important problems arising is construct a proper method of designing controls which transfer a system from a given state to some other preassigned target state. This desired control should usually also satisfy additional requirements: it should be, for example, bounded or optimal in a certain sense.

An important feature of the modern control theory is that the studied problems are mostly nonlinear and, moreover, their linear approximations often lose the structural properties of original systems thus cannot be loosely applied. This means that, among nonlinear systems, the simplest systems should be chosen, in particular, to approximate other nonlinear systems of more complicated structure. This direction is represented by a homogeneous approximation problem which was actively developed during several decades [1, 2, 4, 6, 10, 16, 17].

Along with the differential-geometric methods which are most commonly applied, algebraic methods proved to be useful in such problems. The first idea was given by M. Fliess [8]; he proposed to consider a series in the free associative algebra instead of the control system. One of the follow-ups was proving that the free algebraic interpretation of concepts related to homogeneous approximation allows more exact description of local properties of nonlinear affine control systems [18]. This approach, based mainly on algebraic constructions, is well suitable for numerical implementation.

In this paper we continue this research direction depicting the computational approach to the control synthesis problem of a class of non-linear dynamical systems. Firstly we introduce the needed notions, then we describe the numerical procedures for obtaining homogeneous approximations of our systems. In the end we provide numerical experiments for three systems – two artificially constructed and one physical – illustrating our approach.

2 Theoretical background

Thus, we consider the control system (1), where $X_i(x)$ are real analytic vector fields in the neighborhood of the origin. We are interested in trajectories of this system starting at the origin and corresponding to a control $u(t) = (u_1(t), \ldots, u_m(t))$, which is supposed to be measured and bounded. In this paper we assume that the system is locally controllable in a neighborhood of the origin. Then, for some t_f, the trajectory $x(t)$ of the system exists for $t \in [0, t_f]$. Let us denote by $\mathcal{E}_{X_1,\ldots,X_m}(t_f, u)$ the "end-point map" which takes $u(t)$ to the end point of the trajectory $x(t_f)$. Linearity in the control and analyticity allows expressing the end-point map as a series

$$\mathcal{E}_{X_1,\ldots,X_m}(t_f, u) = \sum_{k=1}^{\infty} \sum_{1 \le i_1,\ldots,i_k \le m} c_{i_1 \ldots i_k} \eta_{i_1 \ldots i_k}(t_f, u), \tag{2}$$

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where $\eta_{i_1\dots i_k}(t_f,u)$ are "iterated integrals" of the form

$$\eta_{i_1\dots i_k}=\eta_{i_1\dots i_k}(t_f,u)=\int_0^{t_f}\int_0^{\tau_1}\cdots\int_0^{\tau_{k-1}}u_{i_1}(\tau_1)\cdots u_{i_k}(\tau_k)d\tau_k\cdots d\tau_1.$$

Coefficients $c_{i_1\dots i_k}\in\mathbb{R}^n$ are defined by the system as

$$c_{i_1\dots i_k}=X_{i_k}X_{i_{k-1}}\cdots X_{i_1}E(0), \tag{3}$$

where $E(x)=x$ is the identity map and X_i are understood as differential operators acting as $X_i\varphi=D\varphi(x)\cdot X_i(x)$, $i=1,\dots,m$.

If controls are such that $|u_i(t)|\le 1$, $t\in[0,t_f]$, iterated integrals satisfy the estimate $|\eta_{i_1\dots i_k}(t_f,u)|\le\frac{1}{k!}t_f^k$. Hence, for small t_f, it is natural to truncate the series in (2) and to consider an approximation that is described by a finite number of iterated integrals. If such a truncated series actually *is* an end-point map for a certain system, then such a system can be regarded as a reasonable approximation of the initial system (1).

It turns out that there exist such coordinates in which one easily finds an appropriate truncation. Suppose we want to write the system (1) in the new coordinates $y=F(x)$; then coefficients of the series for the end-point map in the new coordinates equal $X_{i_k}X_{i_{k-1}}\cdots X_{i_1}F(0)$. This leads to studying the operators $X_{i_k}X_{i_{k-1}}\cdots X_{i_1}$ (so-called nonholonomic derivatives); the goal is to find the convenient *privileged coordinates*: if the system is written in these coordinates then the end-point map takes the componentwise form

$$(\mathcal{E}_{X_1,\dots,X_m})_j(t_f,u)=(\mathcal{E}_{\widehat{X}_1,\dots,\widehat{X}_m})_j(t_f,u)+\rho_j(t_f,u),$$

where $\mathcal{E}_{\widehat{X}_1,\dots,\widehat{X}_m}$ is the end-point map of the system

$$\dot z=\sum_{i=1}^m\widehat{X}_i(z)u_i, \tag{4}$$

which is locally controllable in a neighborhood of the origin and is such that $(\mathcal{E}_{\widehat{X}_1,\dots,\widehat{X}_m})_j$ contains iterated integrals of length w_j and ρ_j contains iterated integrals of length greater than w_j. Then the system (4) is called *a homogeneous approximation* of the system (1). Due to homogeneity (more specifically, quasi-homogeneity), a homogeneous approximation is the simplest approximation of the nonlinear system (1). Numbers w_j are interpreted as weights of coordinates; for convenience assume $w_1\le\dots\le w_n$.

The question is how to construct a homogeneous approximation. In [2], the general method is described, which suggests to find privileged coordinates applying successive polynomial change of coordinates. After a finite number of steps, the system (4) can be constructed. In practical implementation we deal with vector fields, therefore, symbolic computations are needed.

Another way for finding a homogeneous approximation is as follows. Let us start with the series representation (2), where we assume that coefficients $c_{i_1\dots i_k}$ are *fixed vectors*. Then we turn our attention to the iterated integrals.

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As was noticed by M. Fliess [8], for a fixed $t_f > 0$, iterated integrals form a free associative algebra $\mathcal{F}$ over $\mathbb{R}$ with the algebraic operation $\eta_{i_1\dots i_k}\eta_{j_1\dots j_p} = \eta_{i_1\dots i_k j_1\dots j_p}$. We introduce the inner product in $\mathcal{F}$ assuming the basis $\eta_{i_1\dots i_k}$ to be orthonormal. Notice that $\eta_1, \dots, \eta_m$ can be considered as the free generators of $\mathcal{F}$. A natural grading in $\mathcal{F}$ is defined as

$$\mathcal{F} = \sum_{k=1}^{\infty} \mathcal{F}^k, \quad \mathcal{F}^k = \mathrm{Lin}\{\eta_{i_1\dots i_k} : i_1, \dots, i_k \in \{1, \dots m\}\}.$$

We write $\mathrm{ord}(a) = k$ if $a \in \mathcal{F}^k$. Let $\mathcal{L}$ be a free graded Lie algebra generated by $\eta_1, \dots, \eta_m$ with the Lie brackets operation $[\ell_1, \ell_2] = \ell_1\ell_2 - \ell_2\ell_1$ and grading $\mathcal{L} = \sum_{k=1}^{\infty} \mathcal{L}^k$, where $\mathcal{L}^k = \mathcal{L} \cap \mathcal{F}^k$.

Now, with the series (2) we associate a linear map $c : \mathcal{F} \to \mathbb{R}^n$ defined on the basis as $c(\eta_{i_1\dots i_k}) = c_{i_1\dots i_k}$. Suppose an analytic change of variables $y = F(x)$ is applied; then in the new coordinates the end-point map has a series representation $F(\mathcal{E}_{X_1,\dots,X_m})$; it should be written in the form of a series of iterated integrals with vector coefficients. While transforming, products of iterated integrals arise, which should be represented as linear combinations of iterated integrals. In algebraic terms, the product of iterated integrals corresponds to the "shuffle product" defined recursively as

$$\eta_{i_1\dots i_k} \sqcup\!\sqcup \eta_{j_1\dots j_p} = \eta_{i_1}(\eta_{i_2\dots i_k} \sqcup\!\sqcup \eta_{j_1\dots j_p}) + \eta_{j_1}(\eta_{i_1\dots i_k} \sqcup\!\sqcup \eta_{j_2\dots j_p}).$$

The shuffle product in $\mathcal{F}$ and the Lie algebra $\mathcal{L}$ are surprisingly connected: by R. Ree's theorem, an element from $\mathcal{F}$ belongs to $\mathcal{L}$ if and only if it is orthogonal to shuffle product of any two elements from $\mathcal{F}$.

We use this property in order to define the homogeneous approximation. Namely, let us consider the following linear subspace in $\mathcal{L}$,

$$\mathcal{L}_{X_1,\dots,X_m} = \sum_{k=1}^{\infty} \mathcal{P}^k, \text{ where } \mathcal{P}^k = \{\ell \in \mathcal{L}^k : c(\ell) \in c(\mathcal{L}^1 + \dots + \mathcal{L}^{k-1})\}, \ k \geq 1. \tag{5}$$

One can show that $\mathcal{L}_{X_1,\dots,X_m}$ is a graded subalgebra of $\mathcal{L}$ of codimension n; we call it a *core Lie subalgebra* of the system (1). Let $\ell_1, \dots, \ell_n \in \mathcal{L}$ be (homogeneous) elements such that $\mathcal{L} = \mathrm{Lin}\{\ell_1, \dots, \ell_n\} + \mathcal{L}_{X_1,\dots,X_m}$, and let $\{\ell_j\}_{j=n+1}^{\infty}$ be a (homogeneous) basis of $\mathcal{L}_{X_1,\dots,X_m}$. Without loss of generality we assume that $\ell_i \in \mathcal{L}^{w_i}$, $i = 1, \dots, n$, where $w_1 \leq \dots \leq w_n$. Due to the Poincaré-Birkhoff-Witt theorem, elements $\ell_{i_1}^{q_1} \cdots \ell_{i_k}^{q_k}$ form a basis of $\mathcal{F}$, where $i_1 < \dots < i_k$ and ℓ^q means the q-th power of ℓ. Then there exists a biorthogonal basis, and, due to the Melançone-Reutenauer theorem, its elements are of the form

$$\frac{1}{q_1! \cdots q_k!} d_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{i_k}^{\sqcup\!\sqcup q_k},$$

where $d^{\sqcup\!\sqcup q}$ means the q-th shuffle power of d and d_i are elements of the biorthogonal basis orthogonal to all elements of the Poincaré-Birkhoff-Witt basis except ℓ_i and such that the inner product of d_i and ℓ_i equals 1.

The main result about homogeneous approximation is as follows.

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– There exists the system (4) (homogeneous approximation of (1)) whose end-point map equals

$$(\mathcal{E}_{\widehat{X}_1,\dots,\widehat{X}_m})_j = d_j, \quad j = 1,\dots,n. \tag{6}$$

– There exists a change of coordinates $y = F(x)$ in the system (1) which reduces it to the form $\dot{y} = \sum_{i=1}^{m} Y_i(y)u_i$ such that

$$(\mathcal{E}_{Y_1,\dots,Y_m})_j = d_j + \rho_j, \quad j = 1,\dots,n, \tag{7}$$

where ρ_j contains iterated integrals of length greater than w_j.

We emphasize that it may be more convenient to consider the system (4) in other coordinates for which the end-point map equals

$$(\mathcal{E}_{\widehat{X}_1,\dots,\widehat{X}_m})_j = d_j + P_j(d_1,\dots,d_{j-1}), \quad j = 1,\dots,n, \tag{8}$$

where P_j are shuffle polynomials containing elements from $\mathcal{F}^{w_j}$ only.

3 Procedure for obtaining homogeneous approximations

Thus, the rough plan for finding a homogeneous approximation using the results described in the previous section is as follows.

1. Find the core Lie subalgebra and the elements $d_1,\dots,d_n$ of the biorthogonal basis.
2. Reconstruct the system (4) having the end-point map (6).
3. Find a change of variables $y = F(x)$ reducing the end-point map of the system (1) to the form (7).

In [15], the first attempt was taken to implement these steps (for a slightly different class of systems). Now we present a much better implementation, which allows us to perform computer experiments. We discuss the algorithm in more detail.

In step 1, we first of all find the core Lie subalgebra (5). To use its definition, we perform the following steps:

1.1. Find a basis in the free Lie algebra $\mathcal{L}$.
1.2. Find coefficients $c_{i_1\dots i_k}$ of the series (2).
1.3. Find bases for subspaces $\mathcal{P}^k$.

Actually, we need to find $c(b_j)$, where b_j are basis elements in $\mathcal{L}$. Taking into account the grading, we find bases of subspaces $\mathcal{L}^1, \mathcal{L}^2, \dots$ successfully until $\mathcal{L}^r$ such that $c(\mathcal{L}^1 + \dots + \mathcal{L}^r) = \mathbb{R}^n$. More specifically, for any k, we find a basis b_j of $\mathcal{L}^k$ and the coefficients $c_{i_1\dots i_k}$; then we form the vectors $c(b_j)$ and find the subspace $\mathcal{P}^k$.

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Coefficients $c_{i_1\dots i_k}$ are defined by (3). For a sufficiently general form of vector fields X_i, this step is of a large complexity, where symbolic and numerical approaches should be combined. However, some simplification can be implemented: for example, if the coefficient c_j vanishes then all $c_{i_1\dots i_q\, j}$ vanish as well.

Finding a basis in the free Lie algebra is a standard problem. However we observe that the dimension of $\mathcal{L}^k$ grow rapidly, so the complexity here is related to the depth of singularity r rather than the dimension of the system n.

Finally, for step 1 two more sub-steps are needed:

1.4. Find elements of the Poincaré-Birkhoff-Witt basis.
1.5. Find elements of the biorthogonal basis $d_1, \dots, d_n$.

As above, we consider each subspace $\mathcal{F}^k$ separately and restrict ourselves by $k \le r$ and solve linear algebra problems, possible with sparse matrices.

The algorithm for step 2 is given in [18]. Namely, step 2 consists of two sub-steps applied to each of d_j, $j = 1, \dots, n$:

2.1. Represent d_j as a sum $d_j = \sum_{k=1}^{m} \eta_k a_k$. If $w_j = 1$, then $a_k \in \mathbb{R}$. If $w_j \ge 2$, then $a_k \in \mathcal{F}^{w_j - 1}$ and, moreover, a_k equals a shuffle polynomial of $d_1, \dots, d_{j-1}$, i.e.,

$$a_k = \sum \alpha^k_{q_1 \dots q_{j-1}} d_1^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{j-1}^{\sqcup\!\sqcup q_{j-1}}, \text{ where } \alpha^k_{q_1\dots q_{j-1}} \in \mathbb{R}.$$

2.2. For any $k = 1, \dots, n$, define the j-th component of $\widehat{X}_k$ as follows: if $w_j = 1$, then put $(\widehat{X}_k(z))_j = a_k$; if $w_j \ge 2$, then $(\widehat{X}_k(z))_j = \sum \alpha^k_{q_1\dots q_{j-1}} z_1^{q_1} \cdots z_{j-1}^{q_{j-1}}$.

Concerning step 3, we note that such a change of variables is not unique. In particular, one can find $F(x)$ as a polynomial [17,18] in the following way:

3.1. Consider the polynomial mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$\Phi(z) = \sum_{q_1 w_1 + \dots + q_n w_n \le w_n} \frac{1}{q_1! \cdots q_n!} c(\ell_1^{q_1} \cdots \ell_n^{q_n}) z_1^{q_1} \cdots z_n^{q_n}.$$

3.2. Find a mapping $y = F(x)$ that reduces $\Phi(z)$ to the "upper triangular form", i.e., such that component-wise

$$(F(\Phi(z)))_j = z_j + p_j(z_1, \dots, z_{j-1}),$$

where p_j is a polynomial that contains terms of the form $z_1^{q_1} \cdots z_n^{q_n}$ such that $q_1 w_1 + \cdots + q_n w_n \ge w_j + 1$.

Our general goal to provide a fully automatic procedure for any system in the form (1). The presented method is effective, but computational complexity may depend on complexity of the system, of length and order of the endpoint series expansion, and of the number of gaps in the resulting Lie core algebra. Computational issues of the presented procedure consist of combinatorial, symbolic, and numerical algebra procedures. The main difficulty is the need to combine symbolic and numeric computations. There are two separate parts where

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symbolic-based computations are required: high-order differentiation for computations of Lie brackets and Lie elements and finding nonlinear mapping and change of variables. Currently, we use sympy [13] and lambdification, but at least for differentiation, we see the possibility of using modern tools of automatic differentiation like JAX/Autograd [5].

4 Numerical experiments

Systems and transformations. In the experimental part we consider three control systems:

1. Artificial system 1:

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \sin(x_2) \end{pmatrix} u_0 + \begin{pmatrix} x_2 \\ 1 \\ x_1\,(1 - x_3) \\ 1 \end{pmatrix} u_1; \tag{9}$$

2. Artificial system 2:

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - x_1^2 \\ 0 \\ x_3\,(1 - x_0) \end{pmatrix} u_0 + \begin{pmatrix} x_1\,(1 - x_3) \\ 0 \\ x_1\,(1 - x_2) + x_3 \\ 1 \end{pmatrix} u_1; \tag{10}$$

3. Vehicle (truck/lorry) linked to with two trailers:

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \cos(x_2) \\ \sin(x_2) \\ 0 \\ \sin(x_2 - x_3) \\ \sin(x_3 - x_4)\cos(x_2 - x_3) \end{pmatrix} u_0 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u_1, \tag{11}$$

where x_0, x_1 are car position and x_2 is the angle between the car direction and the Ox_0 axis.

For each system, we aim to find the control signal that moves the system from a given initial point to zero.

First, we write the solution of the considered system in the form of the series (2). Using the procedure described in the previous section, we find the transformation that allows the truncation of the series to the homogeneous form.

System (9) has the following homogeneous approximation:

$$\begin{pmatrix} \dot{z}_0 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_0 + \begin{pmatrix} 0 \\ 1 \\ -z_0 z_1 \\ -\frac{1}{6} z_0 z_1^3 \end{pmatrix} u_1, \tag{12}$$

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and the transformation $y = F(x)$ can be chosen as

$$y = F(x) = \begin{pmatrix} x_0 \\ x_1 \\ x_3 - x_1 - \frac{1}{2}x_0x_1^2 \\ \frac{1}{6}x_2 - \frac{1}{12}x_1^2 - \frac{1}{36}x_1^3 - \frac{1}{24}x_0x_1^4 + \frac{1}{12}x_1^2x_3 \end{pmatrix}. \tag{13}$$

Homogeneous approximation of the systems (10) is the following:

$$\begin{pmatrix} \dot{z}_0 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_0 + \begin{pmatrix} 0 \\ 1 \\ z_0 \\ -z_0z_1 \end{pmatrix} u_1, \tag{14}$$

and transformation for the system (10) can be

$$y = F(x) = \begin{pmatrix} x_1 \\ x_3 \\ x_2 - \frac{1}{2}x_3^2 \\ \frac{1}{3}x_0 - \frac{1}{3}x_1 - \frac{1}{3}x_2 - \frac{1}{6}x_1^2 + \frac{1}{6}x_3^2 - \frac{1}{9}x_1^3 - \frac{1}{3}x_1x_3^2 \end{pmatrix}. \tag{15}$$

The homogeneous approximation for the car–trailers system (11) is given by:

$$\begin{pmatrix} \dot{z}_0 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_0 + \begin{pmatrix} 0 \\ 1 \\ z_0 \\ -\frac{1}{2}z_0^2 \\ \frac{1}{6}z_0^3 \end{pmatrix} u_1, \tag{16}$$

and the transformation can be

$$y = F(x) = \begin{pmatrix} x_0 \\ x_2 \\ -x_1 + x_0x_2 \\ -x_4 + x_0x_1 - \frac{1}{2}x_0^2x_2 \\ x_3 - x_1 + x_4 - \frac{1}{2}x_0^2x_1 + x_0x_4 + \frac{1}{6}x_0^3x_2 \end{pmatrix}. \tag{17}$$

As we see, the homogeneous systems (12), (14) and (16) are much simpler than the original systems (9), (10), (11). Please note that the transformations $y = F(x)$ are effectively invertible. The linear part is easy to invert, and the nonlinear part has the feedforward form. As was mentioned above, choosing a homogeneous approximation in a slightly different form (8) could be more convenient, for example, if we are interested in simplifying the change of variables $y = F(x)$.

The homogeneous approximation is a certain kind of simplification of a nonlinear control system, that makes the system easier to being integrated and also allows easier solutions of various controllability problems. A homogeneous approximation simplifies the given system and maintains its main properties like

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controllability. Both systems are equivalent up to a nonlinear transformation. Having the approximation and transformation, we can compare the trajectories of both systems with the same control signals. We can also find the control for the approximated homogeneous system in a simpler transformed space, and we apply the same control to the original system in the transformed space. In order to compare both trajectories, we must transform the homogeneous system trajectory to the original space using inverse transformations. In our experiments we inverse the time in the initial system and in its homogeneous approximation and consider the origin as a goal point.

Results. Comparing the system trajectories with the same control signals, we would like to show and briefly discuss the quality of such approximations. One of objectives of the presented research is to review how good the homogeneous approximation is with regard to the original system.

Finding the optimal control u_i was not the main task in this paper. Nevertheless, because the approximation is valid close to the origin, we should find some control signal for the approximated system that moves the system from a given initial point to the origin (point 0) or close to the origin. In our experiment we chose the sequence of radii of increasing size and randomly selected ten initial points per radius. Each initial point in chosen from an original space is transformed to z's space i.e. $z^0 = F(x^0)$. Next, we try to find a control signal that moves the transformed homogeneous system from the initial point z^0 to the final point 0. Finally, we inspect trajectories of the original and approximated systems in the original space. To transform the trajectory of the homogeneous system to the original space, we have to apply the transformation $x = F^{-1}(z)$. As the quality measure, we use two following measures: the distance between final positions,

$$e_{t_f} = \|x(t_f) - x_h(t_f)\|, \tag{18}$$

and summed up distance between trajectories,

$$e_1 = \int_0^{t_f} \|x(t) - x_h(t)\| dt, \tag{19}$$

where x denotes the original systems' trajectories, and x_h the trajectories of their homogeneous approximations transformed back to the original coordinates.

Detailed results are summarized in Table 1. Figures 2 and 3 present sample trajectories for different initial positions. It is worth to notice that in the car-trailers system (11) our simulation shows that the behaviour of the approximated system is better than the theory suggests. Here the trajectory (namely state variables x_3 and x_4) at some point goes out of the region of applicability but it comes back in later, and in the end the trajectory arrives very close to the end-point of the original system. One can observe this quantitatively in Table 1 where the total (integral) error e_1 may be much larger than e_{t_f} which measures the deviation of the end-point only.

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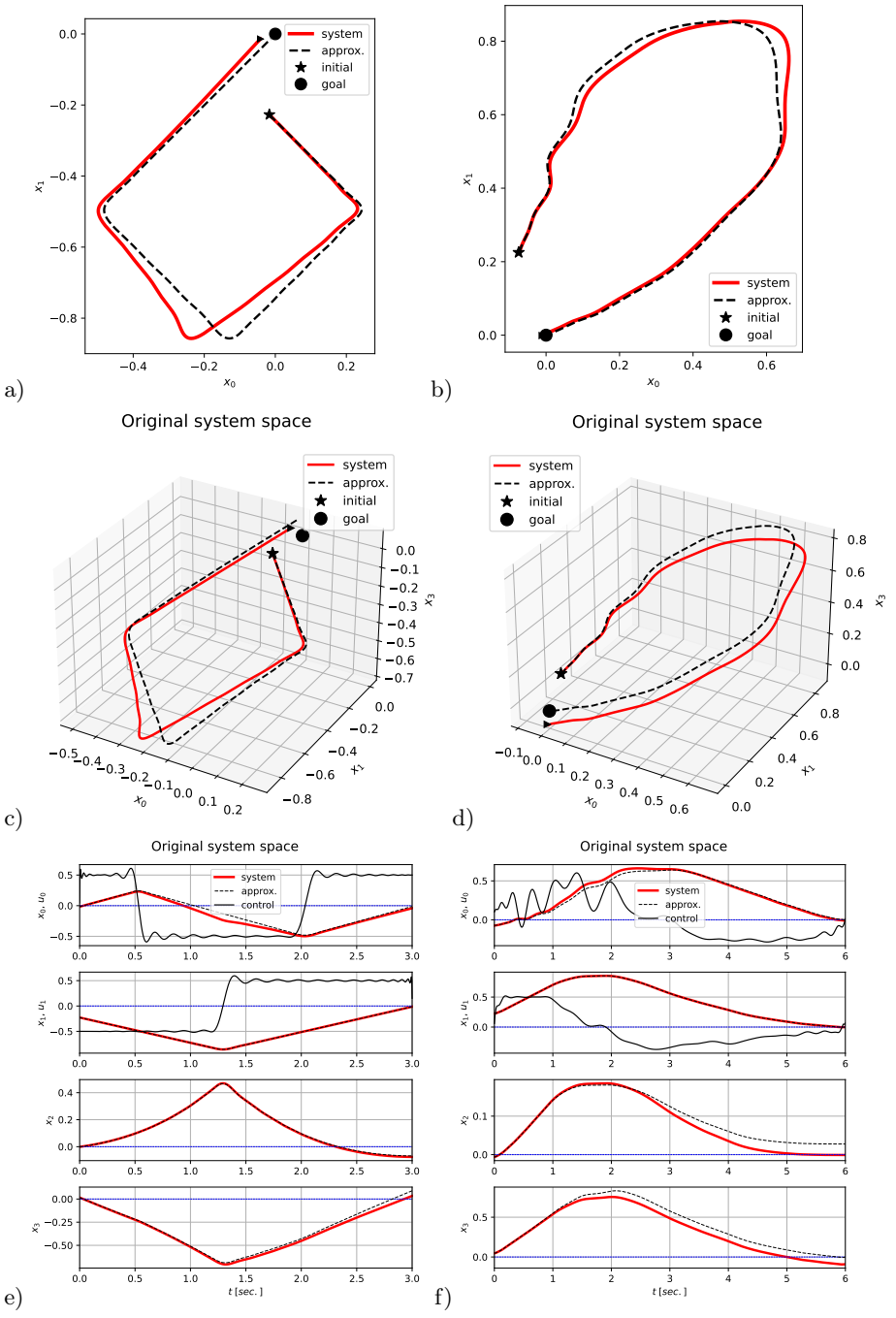

Fig. 1. Illustration of the proposed approach with the system (9); a–d) projection of system's trajectories to 2D and 3D subspaces starting from different initial points. e), f) control signals u_0, u_1 (solid black) and systems' trajectories in separate coordinates. Red solid lines – original system trajectories, black dashed lines represent behavior of approximated system in original system's coordinates

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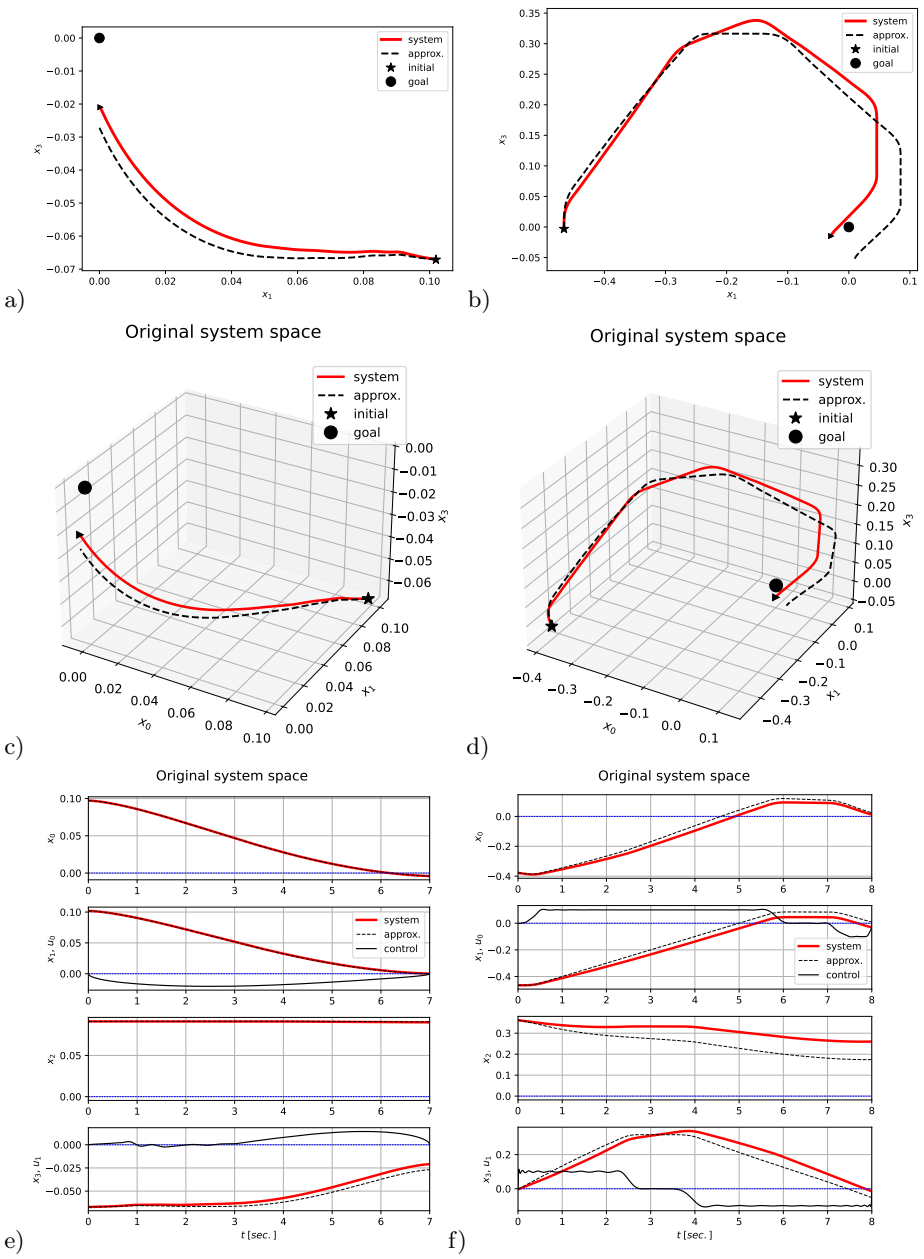

Fig. 2. Illustration of the proposed approach with the system (10); a–d) projection of system's trajectories to 2D and 3D subspaces starting from different initial points. e), f) control signals u_0, u_1 (solid black) and systems' trajectories in separate coordinates. Red solid lines – original system trajectories, black dashed lines represent behavior of approximated system in original system's coordinates

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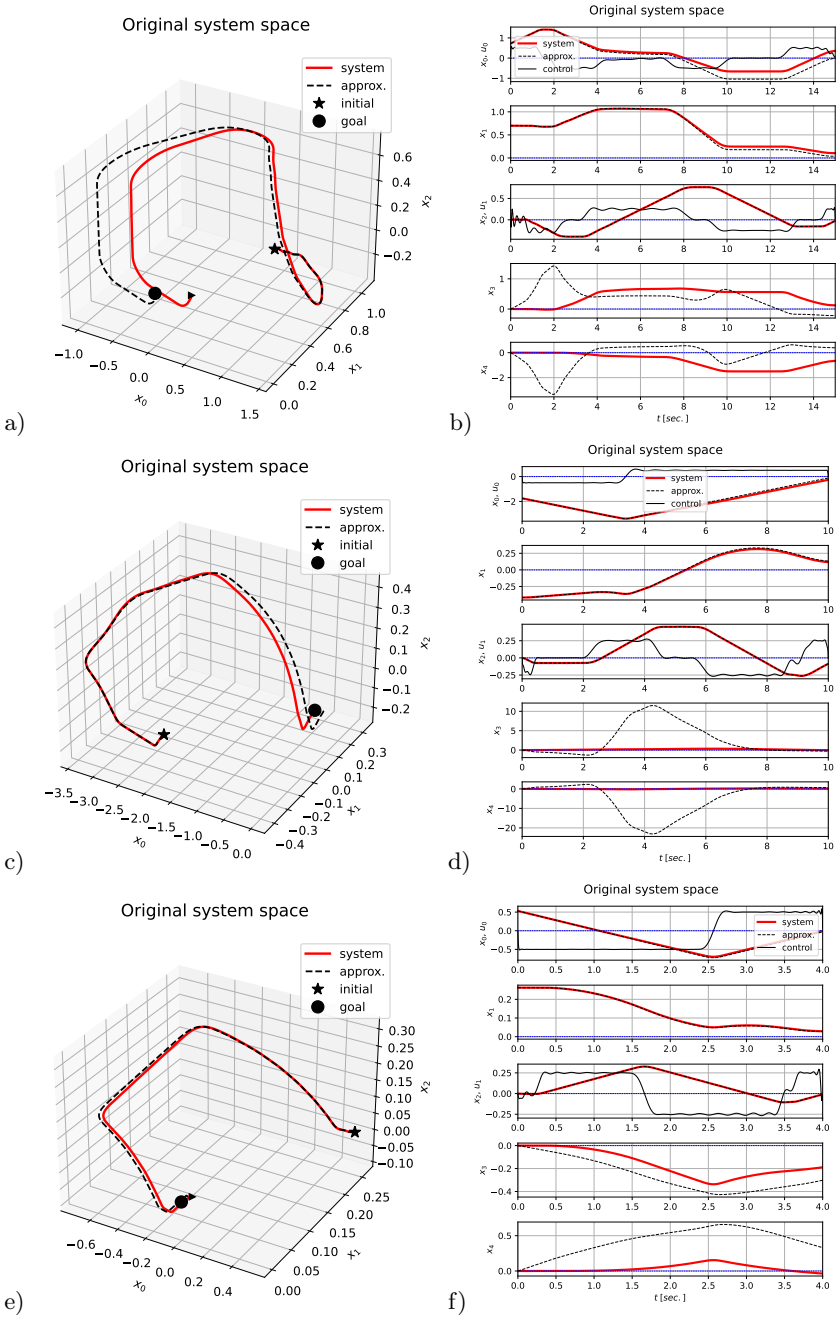

Fig. 3. Illustration of the proposed approach with the car-trailers system (11); a), c), e) projection of system's trajectory to 3D subspace ($x_0 = x$, $x_1 = y$, $x_2 = \alpha$). b), d), f) systems' trajectories in separate coordinates. Black-dashed lines represent behavior of approximated system in original system's coordinates, red lines are original system trajectories. Control signals u_0, u_1 are presented in first and third subplots.

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Numerical details. Now we state some remarks about numerical details of the experiments. A proper numerical representation for control signal and state variables is an important issue. In our experiments we arbitrary choose representations based on Chebyshev polynomials. First, we assume that the control signal is a polynomial of degree n written in the Chebyshev basis of the first kind T_i, $i = 0, \ldots, n$

$$f_n(t) = \frac{1}{2}a_0 + \sum_{i=1}^{n} a_i T_i(t).$$

We do not use it directly, but we use the barycentric interpolation on the Chebyshev points of the second kind (or the Chebyshev extreme points) instead [3]. Interpolation on Chebyshev nodes is equivalent to the Chebyshev expansion. Both representations are widely used in practice, e.g. packages Chebfun [7] and PaCal [11] are based on barycentric interpolation, package numpy.polynomial.chebyshev provides common operations on Chebyshev expansion. The transformation between both representations (called Chebyshev transformation), i.e. transition from the interpolation using Chebyshev nodes to coefficients of expansion in the Chebyshev basis, can be effectively performed using Fast Fourier Transform [12, ch. 4.]. Representation using interpolation is convenient for most operations like algebraic operations of function evaluation. However, the cumulative integration is better to perform using Chebyshev expansion [12, ch. 2.], where

Table 1. Comparison of accuracy of homogeneous approximation. Here $\|X(0)\|$ denotes the norm of initial point, $\|X_h(t_f)\|$ – the norm of the final point (i.e. closeness of the solution to the goal), e_{t_f}, e_1 – global measures of quality of approximation defined by (18) and (19). Each trial represents the average of ten repetitions of the experiment

Model	Trial	$\|X(0)\|$	$\|X_h(t_f)\|$	e_{t_f}	e_1	$e_{t_f}/\|X_0\|$	$e_1/\|X_0\|$
art1	0	0.209	0.082	0.012	0.001	0.061	0.006
	1	0.410	0.095	0.052	0.013	0.128	0.031
	2	0.729	0.296	0.136	0.113	0.163	0.114
	3	1.225	0.134	0.272	0.550	0.187	0.353
art2	0	0.277	0.248	0.008	0.000	0.028	0.001
	1	0.517	0.568	0.036	0.008	0.057	0.011
	2	1.133	0.787	0.162	0.187	0.129	0.141
	3	1.649	1.314	0.634	1.074	0.373	0.608
car	0	0.301	0.172	0.027	0.019	0.088	0.068
	1	0.517	0.400	0.069	0.441	0.125	0.786
	2	1.271	0.411	0.104	28.000	0.094	17.138
	3	1.650	0.377	0.211	9.954	0.133	7.103

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we have a direct formula:

$$\int_0^t f_n(\tau)d\tau = \frac{1}{2}c_0 + \sum_{i=1}^{n} c_i T_i(t),$$

where constant c_0 is determined from the initial condition.

Recalling that the homogeneous system has a special feedforward form, we can easily solve it directly using a presented Chebyshev framework. Under the assumption that control is polynomial in Chebyshev expansion, we use only basic algebraic operations (addition, subtraction and multiplication), cumulative integrals, and feedforward substitutions to solve subsequent equations. Each step of procedure is closed with respect to Chebyshev representation used. Computations with this representation are fast and numerically very stable, accuracy is typically close to the machine precision.

To find the control, we use the optimization procedure with goal function:

$$g(u_{01}, \ldots, u_{1n}; u_{11}, \ldots, u_{1n}) = \|x_h(t_f) - 0\|_2 + \sum_{j=0}^{n} \alpha \|x_j\|_\infty$$

where u_{ij} are values of control signals $u_i, i = 0, 1$ in Chebyshev nodes $t_j, j = 0, 1, \ldots, n_c$ scaled to the interval $[0, t_f]$, t_f is a fixed time of controls, α is a (small) parameter. As the optimization procedure we use l-bfgs-b from scipy [14], with additional restrictions on maximal values of nodes of the control signals.

To compare the quality of the homogeneous approximation we apply the same control signals to the original system. To solve the original system we have to use a general ODE solver. In our case we use odeint function from scipy. For the homogeneous system direct integration using Chebyshev framework is more accurate and much faster. In the experiments we use interpolators with $n_c = 50$ nodes to represent control, in such case scipy's ODE solver was about 4 times slower. For higher degrees this difference will be even bigger.

We prepared experimental part using Python language with numpy [9] libraries mainly for arrays, scipy [19] for optimization and ODE solvers, and sympy [13] for symbolic computing – mainly differentiation and computing of Lie brackets.

5 Summary

In this paper we presented the procedure of determining of a homogeneous approximation from a computational point of view, and we provide the numerical experiments with some nonlinear control systems and their homogeneous approximations. After comparing the system trajectories, we briefly discussed the quality of such approximations. The experiments confirmed that the theoretical results concerning homogeneous approximations of nonlinear systems can be used in practice, and the need to construct suitable software libraries to be used in possible applications – e.g. in control design – is evident.

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