BiWeighted Regular Grid Graphs - A New Class of Graphs for Which Graph Spectral Clustering is Applicable in Analytical Form*

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Abstract. This paper presents a closed form solution to the eigen-problem of combinatorial graph Laplacian for a new type of regular grid graphs - biweighted grid graphs. Biweighted grid graphs differ from ordinary ones in that the weights along a single dimension are altering which adds complexity to the eigen-solutions and makes the graphs better test-bed for potential applications.

Keywords: Artificial Intelligence · Machine Learning · Graph spectral clustering · Analytical solution of eigenvalue problem · Combinatorial Laplacian

1 Introduction

Present day artificial intelligence (AI) is linked tightly to machine learning (ML) solutions, enabling machines to learn from data and subsequently make predictions based on uncovered patterns in data. The ML tools used encompass unsupervised learning (or clustering) methods. One of the intensively developing clustering techniques is Graph Spectral Analysis, encompassing Graph Spectral Clustering (GSC). It works best for objects whose mutual relationships are described by a graph that connects them based on a similarity measure [20, 16, 21]. The concept of Graph Laplacians has been in use for a long time now. An extensive overview of early research can be found in the paper [13] by Merris from the year 1994. In this paper, the author uses *combinatorial Laplacian* $L = D - S$ of a graph G, where S is the adjacency matrix of G, and D is the (diagonal) degree matrix of G. A recent survey can be found in the booklet [8] by Gallier, with a particular orientation towards applications in graph clustering[1].

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[1] For another overview of spectral clustering methods, see e.g. Chapter 5 of the book [21].

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While new GSA algorithms are developed, it is an important issue to have a sufficient amount of test data with clustering properties known in advance. One pathway to this goal is to identify graphs with analytical solutions. One such possibility, although quite simple, is provided by regular grid graphs. Currently, analytical solutions of the eigen-problem of grid Laplacians are known for unweighted grids, which can simulate structures where there are no intrinsic clusters, as well as for weighted grids with different weights in different directions that can simulate structures with known clusters (grid layers separated by links with the lowest weights).

In this paper, we present an analytical solution to the biweighted grid graph, that is one where along one direction the weights are alternating. This structure can be viewed as a better candidate for investigating clustering problems as a cluster can consist not of one but of two layers in a given direction.

The paper is structured as follows. In Section 2, a brief overview of related research is given. In Section 3, our solution to the biweighted grid graph problem is presented. In Section 4, some conclusions are presented.

2 Previous Work

Regular graph structures and their properties are of interest for several reasons, mostly for derivation of analytical graph properties [14]. In particular Ramachandran and Berman [15] exploit a priori knowledge of Laplacians of rectangular grid in investigations of properties of robotic swarms. Stankiewicz [18] discusses relation between the orientable genus of a graph (the minimum number of handles to be added to the plane in order to embed this graph without crossings) and the spectrum of its Laplacian. Cornelissen et al. [3] investigate gonality of curves using grid Laplacians. Merris [13] reviews numerous properties of grid graph Laplacians from the point of view of chemical applications. Cetkovic et al. [4] write about application in mechanics (membrane vibration). They present explicit solutions to the combinatorial Laplacian eigen-problem (eigenvalues and eigenvectors) of the path-graph and as a consequence by the virtue of the construction of the two-dimensional grid graph as a product of path graphs also a solution to the rectangular grid graph combinatorial Laplacian. Cheung et al. [2] elaborate applications in image processing, with a particular interest in grid structures. Burden and Hedstrom [1] were interested in the eigenvalue spectrum of combinatorial Laplacians of grid graphs and derived them from the continuous Laplacian equations. Fiedler [7] established bounds for the second lowest eigenvalue of the combinatorial Laplacian (currently called Fiedler eigenvalue), while mentioning the formula of the Fiedler eigenvalue for the path graph. He also provided a theorem allowing to combine product graph eigenvalues from component graphs. Based on that paper, Anderson and Morey [9] derived explicit formulas for combinatorial Laplacian eigenvalues of grid graphs, without referring to the continuous analogue.

Merris [13] recalls a number of previous results relevant to grid graphs, and also for other special graphs, like tree graphs. Spielman [17] proves explicit for-

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mulas for eigenvalues and eigenvectors for path graphs and grid graphs, without, however, caring about eigenvalues with multiplicity. Fan et al. [6] tackle the issue of signless Laplacians for bicyclic graphs. Edwards [5] found an explicit analytical solution to two-dimensional grid graph Laplacian eigenproblem showing the solution validity in case of eigenvalue ties. Kouachi [10] investigated eigenproperties of tridiagonal matrices, recalling multiple special cases, the topic relevant to path graphs.

The paper [11] presents analytical solutions to normalized and combinatorial Laplacians of grid graphs. The paper [12] investigates the relationship between various types of spectral clustering methods and their kinship to relaxed versions of graph cut methods, based on the closed (or nearly closed) form of eigenvalues and eigenvectors of unnormalized (combinatorial), normalized, and random walk Laplacian of multidimensional weighted and unweighted grids. It is demonstrated the GSA methods can be compared to (normalized) graph cut clustering only if the cut is performed to minimize the sum of weight square roots of removed edges, and not the sum of weights, as generally claimed. In the limit behaviour of combinatorial and normalized Laplacians was investigated showing that the eigenvalues of both converge to one another with increase of the number of nodes while their eigenvectors do not. It is also shown that the distribution of eigenvalues is not uniform in the limit, violating a fundamental assumption of compressive spectral clustering CSC [19].

3 Biweighted Grid Graphs

Let us define a one-dimensional biweighted grid graph as $G_{(n_1)(\mathfrak{w}_1)(\mathfrak{v}_1)}$ being a bi-weighted path graph of n_1 vertices with weight $\mathfrak{w}_1$ for any link in this graph from an odd node i to the even node $i+1$ and with weight $\mathfrak{v}_1$ for any link in this graph from an odd node i to the even node $i-1$. Further, let us define a d-dimensional biweighted grid graph as the weighted graph Cartesian product $G_{(n_1,\dots,n_d)(\mathfrak{w}_1,\dots,\mathfrak{w}_d)(\mathfrak{v}_1,\dots,\mathfrak{v}_d)}$ as $G_{(n_1,\dots,n_{d-1})(\mathfrak{w}_1,\dots,\mathfrak{w}_{d-1})(\mathfrak{v}_1,\dots,\mathfrak{v}_{d-1})} \times G_{(n_d)(\mathfrak{w}_d)(\mathfrak{v}_d)}$ where n_j is the number of layers in the jth dimension and $\mathfrak{w}_j, \mathfrak{v}_j$ are the alternating weights of links between layers in the jth dimension. Integer identities to nodes are assigned as in weighted grid graph.

3.1 Eigensolutions of Combinatorial Laplacians of Bi- weighted Grid Graphs - Path Graph Case

First, let us consider biweighted grid path, which is a one-dimensional graph. The biweighted grid graph treatment, like in the case of weighted grid graph is the product of biweighted grid paths. This means: For an even node e, its entries in the similarity matrix S are of the form $S_{e,e-1} = \mathfrak{w}$, $S_{e,e+1} = \mathfrak{v}$, and all other entries in the row are zeros (the column is accordingly filled). For an odd node o, its entries in the similarity matrix S are of the form $S_{o,o-1} = \mathfrak{v}$, $S_{o,o+1} = \mathfrak{w}$, and all other entries in the row are zeros (the column is accordingly filled). Therefore the Laplacian entries for an even non-border node e are of

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the form: $L_{e,e-1} = -\mathfrak{w}$,$L_{e,e} = \mathfrak{w} + \mathfrak{v}$, $L_{e,e+1} = -\mathfrak{v}$. The Laplacian entries for an odd non-border node o are of the form: $L_{o,o-1} = -\mathfrak{v}$, $L_{o,o} = \mathfrak{w} + \mathfrak{v}$, $L_{o,o+1} = -\mathfrak{w}$. The Laplacian entries for the first (hence border odd) node 1 are of the form: $L_{1,1} = \mathfrak{w}$, $L_{e,e+1} = -\mathfrak{w}$. The last node can be either even or odd. The Laplacian entries for the last even border node l_e are of the form: $L_{le,le-1} = -\mathfrak{w}$,$L_{le,le} = \mathfrak{w}$. The Laplacian entries for the last odd border node l_o are of the form: $L_{o,o-1} = -\mathfrak{v}$, $L_{o,o} = \mathfrak{v}$.

Other entries in the respective rows are zeros. Column entries follow the symmetry principle of L.

Let us illustrate the biweighted graph path with a small example, where $n = 5$, $\mathfrak{w} = 2, \mathfrak{v} = 3$. See Figure 1.

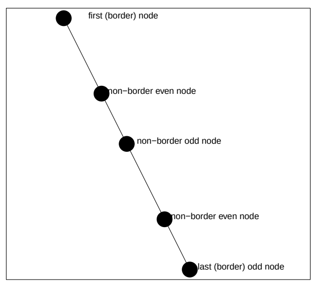

Fig. 1. An example of a biweighted path graph. Shorter edges illustrate higher edge weight ($\mathfrak{v} = 3$) and longer edges lower weight ($\mathfrak{w} = 2$)

The similarity matrix S and its combinatorial Laplacian L have the form

$$S = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 5 & -3 & 0 & 0 \\ 0 & -3 & 5 & -2 & 0 \\ 0 & 0 & -2 & 5 & -3 \\ 0 & 0 & 0 & -3 & 3 \end{pmatrix}$$

Let us introduce some notation: Let n be the number of nodes on the path. If $\mathbf{v}$ is an eigenvector of L, then $\mathbf{v}' = L\mathbf{v}$. Let $\lambda_{s[z]}, \lambda_{c[z]}$ be the eigenvalues that we seek, where $z = 1, \dots n$. The lower indexes s, c indicate which type of eigenvectors will be used, based on the sine (in case of s, see eq. (8,9)) or cosine function (in case of c, see eq. (5,6)). Let $\boldsymbol{\nu}_{s[z]}, \boldsymbol{\nu}_{c[z]}$ be the corresponding eigenvectors. Let $\nu_{s[z],[x]}, \nu_{c[z],[x]}$ with $x = 1, \dots, n$ be the n components of the zth eigenvector $\boldsymbol{\nu}_{[\boldsymbol{z}]}$.

Let

$$\delta_{\mathfrak{w}\mathfrak{v},[z]} = \frac{z2\pi}{n} \tag{1}$$

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Let, if $\mathfrak{w} \geq \mathfrak{v}$ then

$$\delta_{\mathfrak{w},[z]} = \arctan \frac{\mathfrak{v} \sin(\delta_{\mathfrak{wv},[z]})}{\mathfrak{v} \cos(\delta_{\mathfrak{wv},[z]}) + \mathfrak{w}} \tag{2}$$

else

$$\delta_{\mathfrak{v},[z]} = \arctan \frac{\mathfrak{w} \sin(\delta_{\mathfrak{wv},[z]})}{\mathfrak{w} \cos(\delta_{\mathfrak{wv},[z]}) + \mathfrak{v}} \tag{3}$$

whereby the result of arctan is taken from the interval from the range $(-\frac{\pi}{2}, \frac{\pi}{2})$ so that the sine of that δ has the same sign as the sine of the right-hand side, whereby the other δ is computed from the relationship $\delta_{\mathfrak{wv},[z]} = \delta_{\mathfrak{w},[z]} + \delta_{\mathfrak{v},[z]}$.

We will prove in this section the following theorem.

Theorem 1. *The analytical solution of the eigenproblem for biweighted path graphs is of the following form.*

$$\lambda_{c[z]} = \mathfrak{w}(1 - \cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1 - \cos(\delta_{\mathfrak{v}})) \tag{4}$$

$$\nu_{c[z],[2x+1]} = \cos(\frac{1}{2}\delta_{\mathfrak{v}} + x\delta_{\mathfrak{wv}}) \tag{5}$$

$$\nu_{c[z],[2x]} = \cos(\frac{1}{2}\delta_{\mathfrak{v}} + (x-1)\delta_{\mathfrak{wv}} + \delta_{\mathfrak{w}}) \tag{6}$$

$$\lambda_{s[z]} = \mathfrak{w}(1 + \cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1 + \cos(\delta_{\mathfrak{v}})) \tag{7}$$

$$\nu_{s[z],[2x+1]} = -(-1)^{nodeid} \sin(\frac{1}{2}\delta_{\mathfrak{v}} + x\delta_{\mathfrak{wv}}) \tag{8}$$

$$\nu_{s[z],[2x]} = -(-1)^{nodeid} \sin(\frac{1}{2}\delta_{\mathfrak{v}} + x\delta_{\mathfrak{wv}} + \delta_{\mathfrak{w}}) \tag{9}$$

for $x = 0, 1, 2, \ldots$.

If n is even, then $z = 1 : n/2$ first $\lambda_{s[z]}$ eigenvalues and corresponding eigenvectors and first $z = 1 : n/2 - 1$ and $z = n$ $\lambda_{c[z]}$ eigenvalues and corresponding eigenvectors are taken. If n is odd, then $z = 1 : (n-1)/2$ first $\lambda_{s[z]}$ elements and $z = 1 : (n-1)/2$ and $z = n$ $\lambda_{c[z]}$ elements are taken. If $z = n$, then $\lambda_{c[z]}$ is equal zero and $\boldsymbol{\nu}_{c[z]}$ is a constant vector. [2]

Subsequently, we will generally omit the z index unless it turns out to be necessary.

In the above example, the eigenvalues implied by Theorem 1 are:
$0.000, 0.912, 3.186, 6.814, 9.088$.
For eigenvalue 3.186 we have eigenvector [0.939 -0.556 -0.962 -0.038 0.618]. For eigenvalue 9.088 we have eigenvector [0.240 -0.849 0.997 -0.765 0.377]. For eigenvalue 6.814 we have eigenvector [-0.345 0.831 -0.272 -0.999 0.786]. For eigenvalue 0.912 we have eigenvector [-0.971 -0.528 -0.072 0.645 0.926]. For eigenvalue 0 we have eigenvector [1 1 1 1 1].

[2] If n is even and $z = n/2$, then $\boldsymbol{\nu}_{c[z]}$ is a zero vector and therefore respective $\lambda_{c[z]}$ is not used as a solution.

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3.2 Eigensolutions of Combinatorial Laplacians of Bi- weighted Grid Graphs - Path Graph Case with Cosine Shaped Functions

As visible from eq. (5) and (6), our working hypothesis is that the eigenvector $\mathbf{v}$ elements are of the form

$$v_{nodeid} = \cos(\alpha_{nodeid}) \tag{10}$$

and the angles α_{nodeid} differ between neighbouring nodes by alternating either $\delta_{\mathfrak{w}}$ or $\delta_{\mathfrak{v}}$, when we have to do with a path graph.

Non-border even nodes For an even node e, its eigenvector component amounts to $v_e = \cos(\alpha_e)$, its preceding (odd) node component is $v_{e-1} = \cos(\alpha_e - \delta_{\mathfrak{w}})$ its succeeding (odd) node component is $v_{e+1} = \cos(\alpha_e + \delta_{\mathfrak{v}})$.

Upon multiplication of the Laplacian matrix with the eigenvector the result for a non-border even e node would be

$$\begin{aligned} v'_e =& - \mathfrak{w}\cos(\alpha_e - \delta_{\mathfrak{w}}) + (\mathfrak{w}+\mathfrak{v})\cos(\alpha_e) - \mathfrak{v}\cos(\alpha_e + \delta_{\mathfrak{v}}) \\ =&\mathfrak{w}(\cos(\alpha_e) - \cos(\alpha_e - \delta_{\mathfrak{w}})) + \mathfrak{v}(\cos(\alpha_e) - \cos(\alpha_e + \delta_{\mathfrak{v}})) \\ =&\mathfrak{w}(\cos(\alpha_e) - \cos(\alpha_e)\cos(\delta_{\mathfrak{w}}) - \sin(\alpha_e)\sin(\delta_{\mathfrak{w}})) \\ &+ \mathfrak{v}(\cos(\alpha_e) - \cos(\alpha_e) cos(\delta_{\mathfrak{v}}) + \sin(\alpha_e)\sin(\delta_{\mathfrak{v}})) \\ =& \cos(\alpha_e)(\mathfrak{w}(1-\cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1-\cos(\delta_{\mathfrak{v}}))) - \sin(\alpha_e)(\mathfrak{w}\sin(\delta_{\mathfrak{w}}) - \mathfrak{v}\sin(\delta_{\mathfrak{v}})) \end{aligned}$$

If we assume that

$$-\mathfrak{v}\sin(\delta_{\mathfrak{v}}) + \mathfrak{w}\sin(\delta_{\mathfrak{w}}) = 0 \tag{11}$$

then

$$v'_e = \cos(\alpha_e)(\mathfrak{w}(1-\cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1-\cos(\delta_{\mathfrak{v}}))) \tag{12}$$

Recall that the hypothesised eigenvector component v_e is of the form $\cos(\alpha_e)$. So the eigenvalue requirement is fulfilled here with the constant factor $\lambda = \mathfrak{w}(1-\cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1-\cos(\delta_{\mathfrak{v}}))$, as $\mathfrak{v}, \mathfrak{w}, \delta_{\mathfrak{v}}, \delta_{\mathfrak{w}}$ are constants. This fits eq. (4).

Non-border odd nodes For an odd node o, its eigenvector component amounts to $v_o = \cos(\alpha_o)$, its preceding (even) node component is $v_{o-1} = \cos(\alpha_o - \delta_{\mathfrak{v}})$ its succeeding (odd) node component is $v_{o+1} = \cos(\alpha_o + \delta_{\mathfrak{w}})$.

Hence upon multiplication of the Laplacian matrix with the eigenvector the result for a non-border odd o node would be

$$v'_o = -\mathfrak{v}\cos(\alpha_o - \delta_{\mathfrak{v}}) + (\mathfrak{w}+\mathfrak{v})\cos(\alpha_o) - \mathfrak{w}\cos(\alpha_o + \delta_{\mathfrak{w}})$$

That is

$$\begin{aligned} v'_o =&\mathfrak{v}(\cos(\alpha_o) - \cos(\alpha_o - \delta_{\mathfrak{v}})) + \mathfrak{w}(\cos(\alpha_o) - \cos(\alpha_o + \delta_{\mathfrak{w}})) \\ =&\mathfrak{v}(\cos(\alpha_o) - \cos(\alpha_o)\cos(\delta_{\mathfrak{v}}) - \sin(\alpha_o)\sin(\delta_{\mathfrak{v}})) \\ &+ \mathfrak{w}(\cos(\alpha_o) - \cos(\alpha_o)\cos(\delta_{\mathfrak{w}}) + \sin(\alpha_o)\sin(\delta_{\mathfrak{w}})) \\ =& \cos(\alpha_o)(\mathfrak{v}(1-\cos(\delta_{\mathfrak{v}})) + \mathfrak{w}(1-\cos(\delta_{\mathfrak{w}}))) - \sin(\alpha_o)(\mathfrak{v}\sin(\delta_{\mathfrak{v}}) - \mathfrak{w}\sin(\delta_{\mathfrak{w}})) \end{aligned}$$

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Recall that the hypothesised eigenvector component v_o is of the form $\cos(\alpha_o)$. As $\mathfrak{w}, \mathfrak{v}, \delta_{\mathfrak{w}}, \delta_{\mathfrak{v}}$ are constants, the condition of $\mathbf{v}$ being an eigenvector requires that $-\mathfrak{v}\sin(\delta_{\mathfrak{v}}) + \mathfrak{w}\sin(\delta_{\mathfrak{w}}) = 0$, as previously in eq. (11). Then

$$v'_o = \cos(\alpha_o)(\mathfrak{v}(1 - \cos(\delta_{\mathfrak{v}})) + \mathfrak{w}(1 - \cos(\delta_{\mathfrak{w}}))) \tag{13}$$

and the λ factor is the same as above.

First border node We need now to discuss the behaviour of the border nodes. It is an odd node, but without a preceding node.

$$v'_1 = \mathfrak{w}\cos(\alpha_1) - \mathfrak{w}\cos(\alpha_1 + \delta_{\mathfrak{w}}) = \mathfrak{w}(\cos(\alpha_1) - \cos(\alpha_1 + \delta_{\mathfrak{w}}))$$

Let us guess that $\alpha_1 = \frac{1}{2}\delta_{\mathfrak{v}}$ (eq. (5)) . Under this assumption: $\cos(\alpha_1 - \delta_{\mathfrak{v}}) = \cos(\frac{1}{2}\delta_{\mathfrak{v}} - \delta_{\mathfrak{v}}) = \cos(-\frac{1}{2}\delta_{\mathfrak{v}}) = \cos(\frac{1}{2}\delta_{\mathfrak{v}}) = \cos(\alpha_1)$. Hence

$$\begin{aligned} v'_1 =& \mathfrak{v}(\cos(\alpha_1) - \cos(\alpha_1 - \delta_{\mathfrak{v}})) + \mathfrak{w}(\cos(\alpha_1) - \cos(\alpha_1 + \delta_{\mathfrak{w}})) \\ =& \mathfrak{v}(\cos(\alpha_1) - \cos(\alpha_1)\cos(\delta_{\mathfrak{v}}) - \sin(\alpha_1)\sin(\delta_{\mathfrak{v}})) \\ &+ \mathfrak{w}(\cos(\alpha_1) - \cos(\alpha_1)\cos(\delta_{\mathfrak{w}}) + \sin(\alpha_1)\sin(\delta_{\mathfrak{w}})) \\ =& \cos(\alpha_1)(\mathfrak{v}(1 - \cos(\delta_{\mathfrak{v}})) + \mathfrak{w}(1 - \cos(\delta_{\mathfrak{w}}))) - \sin(\alpha_1)(\mathfrak{v}\sin(\delta_{\mathfrak{v}}) - \mathfrak{w}\sin(\delta_{\mathfrak{w}})) \end{aligned}$$

Assuming again that $-\mathfrak{v}\sin(\delta_{\mathfrak{v}}) + \mathfrak{w}\sin(\delta_{\mathfrak{w}}) = 0$, then the eigenvalue requirement is fulfilled here with the same constant (λ) factor $(\mathfrak{w}(1-\cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1-\cos(\delta_{\mathfrak{v}})))$.

Last even border node So consider now the last node when the number of nodes is even. It can be either even or odd. Upon multiplication of the Laplacian matrix with the eigenvector the result for the last border even l_e node would be

$$v'_{le} = -\mathfrak{w}\cos(\alpha_{le} - \delta_{\mathfrak{w}}) + \mathfrak{w}\cos(\alpha_{le}) = \mathfrak{w}(\cos(\alpha_{le}) - \cos(\alpha_{le} - \delta_{\mathfrak{w}}))$$

Let us assume $\alpha_{le} = -\frac{1}{2}\delta_{\mathfrak{v}}$ (eq. (6) in which case $v_{le} = \cos(\frac{1}{2}\delta_{\mathfrak{v}})$. Then clearly $\cos(\alpha_{le} + \delta_{\mathfrak{v}}) = \cos(-\frac{1}{2}\delta_{\mathfrak{v}} + \delta_{\mathfrak{v}}) = \cos(\frac{1}{2}\delta_{\mathfrak{v}}) = \cos(\alpha_{le})$. Therefore

$$\begin{aligned} v'_{le} =& \mathfrak{v}(\cos(\alpha_{le}) - \cos(\alpha_{le} + \delta_{\mathfrak{v}})) + \mathfrak{w}(\cos(\alpha_{le}) - \cos(\alpha_{le} - \delta_{\mathfrak{w}})) \\ =& \mathfrak{v}(\cos(\alpha_{le}) - \cos(\alpha_{le})\cos(\delta_{\mathfrak{v}}) + \sin(\alpha_{le})\sin(\delta_{\mathfrak{v}})) \\ &+ \mathfrak{w}(\cos(\alpha_{le}) - \cos(\alpha_{le})\cos(\delta_{\mathfrak{w}}) - \sin(\alpha_{le})\sin(\delta_{\mathfrak{w}})) \\ =& \cos(\alpha_{le})(\mathfrak{v}(1 - \cos(\delta_{\mathfrak{v}})) + \mathfrak{w}(1 - \cos(\delta_{\mathfrak{w}}))) + \sin(\alpha_{le})(\mathfrak{v}\sin(\delta_{\mathfrak{v}}) - \mathfrak{w}\sin(\delta_{\mathfrak{w}})) \end{aligned}$$

Assuming again that $-\mathfrak{v}\sin(\delta_{\mathfrak{v}}) + \mathfrak{w}\sin(\delta_{\mathfrak{w}}) = 0$, then the eigenvalue requirement is fulfilled here with the same constant (λ) factor $(\mathfrak{w}(1-\cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1-\cos(\delta_{\mathfrak{v}})))$. Let us assume now that $\alpha_{le} = -\frac{1}{2}\delta_{\mathfrak{v}} + \pi$ in which case $v_{le} = -\cos(\frac{1}{2}\delta_{\mathfrak{v}})$. Then clearly $\cos(\alpha_{le}+\delta_{\mathfrak{v}}) = \cos(-\frac{1}{2}\delta_{\mathfrak{v}}+\pi+\delta_{\mathfrak{v}}) = \cos(\frac{1}{2}\delta_{\mathfrak{v}}+\pi) = -\cos(\frac{1}{2}\delta_{\mathfrak{v}}) = \cos(\alpha_{le})$. By the same derivation as above, we find that the eigenvalue requirement is satisfied.

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Last odd border node So consider now the last node, when the number of nodes is odd. It can be either even or odd. Upon multiplication of the Laplacian matrix with the eigenvector the result for the last border even l_o node would be

$$v'_{lo} = -\mathfrak{v}\cos(\alpha_{lo} - \delta_{\mathfrak{v}}) + \mathfrak{v}\cos(\alpha_{lo}) = \mathfrak{v}(\cos(\alpha_{lo}) - \cos(\alpha_{lo} - \delta_{\mathfrak{v}}))$$

Let us assume $\alpha_{lo} = -\frac{1}{2}\delta_{\mathfrak{w}}$ (eq. (5)) in which case $v_{lo} = \cos(\frac{1}{2}\delta_{\mathfrak{w}})$. Then clearly $\cos(\alpha_{lo} + \delta_{\mathfrak{w}}) = \cos(-\frac{1}{2}\delta_{\mathfrak{w}} + \delta_{\mathfrak{w}}) = \cos(\frac{1}{2}\delta_{\mathfrak{w}}) = \cos(\alpha_{lo})$. Therefore

$$\begin{aligned} v'_{lo} =& \mathfrak{w}(\cos(\alpha_{lo}) - \cos(\alpha_{lo} + \delta_{\mathfrak{w}})) + \mathfrak{v}(\cos(\alpha_{lo}) - \cos(\alpha_{lo} - \delta_{\mathfrak{v}})) \\ =& \mathfrak{w}(\cos(\alpha_{lo}) - \cos(\alpha_{lo})\cos(\delta_{\mathfrak{w}}) + \sin(\alpha_{lo})\sin(\delta_{\mathfrak{w}})) \\ &+ \mathfrak{v}(\cos(\alpha_{lo}) - \cos(\alpha_{lo})\cos(\delta_{\mathfrak{v}}) - \sin(\alpha_{lo})\sin(\delta_{\mathfrak{v}})) \\ =& \cos(\alpha_{lo})(\mathfrak{w}(1 - \cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1 - \cos(\delta_{\mathfrak{v}}))) + \sin(\alpha_{lo})(\mathfrak{w}\sin(\delta_{\mathfrak{w}}) - \mathfrak{v}\sin(\delta_{\mathfrak{v}})) \end{aligned}$$

Assuming again that $-\mathfrak{w}\sin(\delta_{\mathfrak{w}}) + \mathfrak{v}\sin(\delta_{\mathfrak{v}}) = 0$, then the eigenvalue requirement is fulfilled here with the same constant (λ) factor $(\mathfrak{v}(1-\cos(\delta_{\mathfrak{v}})) + \mathfrak{w}(1-\cos(\delta_{\mathfrak{w}})))$. Let us assume now that $\alpha_{le} = -\frac{1}{2}\delta_{\mathfrak{w}} + \pi$ in which case $v_{le} = -\cos(\frac{1}{2}\delta_{\mathfrak{w}})$. Then clearly $\cos(\alpha_{le} + \delta_{\mathfrak{w}}) = \cos(-\frac{1}{2}\delta_{\mathfrak{w}} + \pi + \delta_{\mathfrak{w}}) = \cos(\frac{1}{2}\delta_{\mathfrak{w}} + \pi) = -\cos(\frac{1}{2}\delta_{\mathfrak{w}}) = \cos(\alpha_{le})$. By the same derivation as previously, we find that the eigenvalue requirement is satisfied.

3.3 Eigensolutions of Combinatorial Laplacians of Bi- weighted Grid Graphs - Path Graph Case with Sine Shaped Function

Our working hypothesis is that the eigenvector $\mathbf{v}$ elements are of the form

$$v_{nodeid} = -(-1)^{nodeid}\sin(\alpha_{nodeid}) \tag{14}$$

and the angles α_{nodeid} differ between neighbouring nodes by either $\delta_{\mathfrak{w}}$ or $\delta_{\mathfrak{v}}$, when we have to do with a path graph (eq. (8)).

Non-border even nodes For an even node e, its eigenvector component amounts to $v_e = -(-1)^e\sin(\alpha_e)$, its preceding (odd) node component is $v_{e-1} = -(-1)^{e-1}\sin(\alpha_e - \delta_{\mathfrak{w}})$ its succeeding (odd) node component is $v_{e+1} = -(-1)^{e+1}\sin(\alpha_e + \delta_{\mathfrak{v}})$. Upon multiplication of the Laplacian matrix with the eigenvector the result for a non-border even e node would be

$$\begin{aligned} v'_e =& \mathfrak{w}(-1)^{e-1}\sin(\alpha_e - \delta_{\mathfrak{w}}) - (\mathfrak{w} + \mathfrak{v})(-1)^e\sin(\alpha_e) + \mathfrak{v}(-1)^{e+1}\sin(\alpha_e + \delta_{\mathfrak{v}}) \\ =& -\mathfrak{w}\sin(\alpha_e - \delta_{\mathfrak{w}}) - (\mathfrak{w} + \mathfrak{v})\sin(\alpha_e) - \mathfrak{v}\sin(\alpha_e + \delta_{\mathfrak{v}}) \\ =& -\mathfrak{w}(\sin(\alpha_e) + \sin(\alpha_e - \delta_{\mathfrak{w}})) - \mathfrak{v}(\sin(\alpha_e) + \sin(\alpha_e + \delta_{\mathfrak{v}})) \\ =& -\mathfrak{w}(\sin(\alpha_e) + \sin(\alpha_e)\cos(\delta_{\mathfrak{w}}) - \cos(\alpha_e)sin(\delta_{\mathfrak{w}})) \\ &- \mathfrak{v}(\sin(\alpha_e) + \sin(\alpha_e)cos(\delta_{\mathfrak{v}}) + \cos(\alpha_e)\sin(\delta_{\mathfrak{v}})) \\ =& -\sin(\alpha_e)(\mathfrak{w}(1 + \cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1 + cos(\delta_{\mathfrak{v}}))) + cos(\alpha_e)(\mathfrak{w}sin(\delta_{\mathfrak{w}}) - \mathfrak{v}\sin(\delta_{\mathfrak{v}})) \end{aligned}$$

If we assume that $-\mathfrak{v}\sin(\delta_{\mathfrak{v}}) + \mathfrak{w}\sin(\delta_{\mathfrak{w}}) = 0$, then

$$v'_e = -\sin(\alpha_e)(\mathfrak{w}(1 + \cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1 + cos(\delta_{\mathfrak{v}})))$$

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The hypothesised eigenvector component v_e is of the form $-(-1)^e \sin(\alpha_e) = -\sin(\alpha_e)$. So the eigenvalue requirement is fulfilled here with the constant factor $\lambda = \mathfrak{w}(1 + \cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1 + \cos(\delta_{\mathfrak{v}}))$, as $\mathfrak{v}, \mathfrak{w}, \delta_{\mathfrak{v}}, \delta_{\mathfrak{w}}$ are constants.

Non-border odd nodes For an odd node o, its eigenvector component amounts to $v_o = -(-1)^o \sin(\alpha_o)$, its preceding (even) node component is $v_{o-1} = -(-1)^{o-1} \sin(\alpha_o - \delta_{\mathfrak{v}})$ its succeeding (odd) node component is $v_{o+1} = -(-1)^{o+1} \sin(\alpha_o + \delta_{\mathfrak{w}})$. Hence upon multiplication of the Laplacian matrix with the eigenvector the result for a non-border odd o node would be

$$\begin{aligned} v'_o =& \mathfrak{v}(-1)^{o-1} \sin(\alpha_o - \delta_{\mathfrak{v}}) - (\mathfrak{w} + \mathfrak{v})(-1)^o \sin(\alpha_o) + \mathfrak{w}(-1)^{o+1} \sin(\alpha_o + \delta_{\mathfrak{w}}) \\ =& \sin(\alpha_o)(\mathfrak{v}(1 + cos(\delta_{\mathfrak{v}})) + \mathfrak{w}(1 + cos(\delta_{\mathfrak{w}}))) + cos(\alpha_o)(-\mathfrak{v} \sin(\delta_{\mathfrak{v}}) + \mathfrak{w} \sin(\delta_{\mathfrak{w}})) \end{aligned}$$

The hypothesised eigenvector component v_o is of the form $-(-1)^o \sin(\alpha_o) = \sin(\alpha_o)$. As $\mathfrak{w}, \mathfrak{v}, \delta_{\mathfrak{w}}, \delta_{\mathfrak{v}}$ are constants, the condition of $\mathbf{v}$ being an eigenvector requires that $-\mathfrak{v} \sin(\delta_{\mathfrak{v}}) + \mathfrak{w} \sin(\delta_{\mathfrak{w}}) = 0$, as previously. Then

$$v'_o = \sin(\alpha_o)(\mathfrak{v}(1 + cos(\delta_{\mathfrak{v}})) + \mathfrak{w}(1 + cos(\delta_{\mathfrak{w}})))$$

and the λ factor is the same as above.

First border node We need now to discuss the behaviour of the border nodes. In order to ensure that also the eigen-property holds at the end points of the path, we have to take $\alpha_1 = \frac{1}{2}\delta_{\mathfrak{v}}$ for the first node. because in this case the result of the product with the Laplacian matrix ($L_{1,1} = \mathfrak{w}$, $L_{1,2} = -\mathfrak{w}$) with the eigenvector $\mathbf{v}$ at v_1 will amount to:

$$\begin{aligned} v'_1 =& -(-1)^1 \sin(\alpha_1)\mathfrak{w} - (-1)^1 \sin(\alpha_1 + \delta_{\mathfrak{w}})(-\mathfrak{w}) \\ =& \sin(\frac{1}{2}\delta_{\mathfrak{v}})(\mathfrak{w}(1 + \cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1 + \cos(\delta_{\mathfrak{v}}))) + \cos(\frac{1}{2}\delta_{\mathfrak{v}})(\mathfrak{w} \sin(\delta_{\mathfrak{w}} - \mathfrak{v} \sin(\delta_{\mathfrak{v}})) \end{aligned}$$

Recall that $v_1 = -\sin(\frac{1}{2}\delta_{\mathfrak{v}})$ Assuming again that $-\mathfrak{v} \sin(\delta_{\mathfrak{v}}) + \mathfrak{w} \sin(\delta_{\mathfrak{w}}) = 0$, then the eigenvalue requirement is fulfilled here with the same constant (λ) factor $(\mathfrak{w}(1 + \cos(\delta_{\mathfrak{w}}) + \mathfrak{v}(1 + cos(\delta_{\mathfrak{v}})))$.

Last odd border node For an odd border nodel_o, let us take $\alpha_{lo} = -\frac{1}{2}\delta_{\mathfrak{w}}$. Its eigenvector component amounts to $v_{lo} = -(-1)^{l_o} \sin(\alpha_{lo}) = \sin(\alpha_{lo})$, its preceding (even) node component is $v_{l_o-1} = -(-1)^{l_o-1} \sin(\alpha_{lo} - \delta_{\mathfrak{v}})$ Hence upon multiplication of the Laplacian matrix with the eigenvector the result for a last odd l_o node would be

$$\begin{aligned} v'_{lo} =& \mathfrak{v}(-1)^{l_o-1} \sin(\alpha_{l_o} - \delta_{\mathfrak{v}}) - \mathfrak{v}(-1)^{l_o} \sin(\alpha_{l_o}) \\ =& \mathfrak{v} \sin(\alpha_{l_o} - \delta_{\mathfrak{v}}) + \mathfrak{v} \sin(\alpha_{l_o}) = \mathfrak{v}(\sin(\alpha_{l_o} - \delta_{\mathfrak{v}}) + \sin(\alpha_{l_o})) \end{aligned}$$

Let us assume $\alpha_{lo} = -\frac{1}{2}\delta_{\mathfrak{w}}$ in which case $v_{lo} = \sin(\frac{1}{2}\delta_{\mathfrak{w}})$. Then clearly $\sin(\alpha_{lo} + \delta_{\mathfrak{w}}) = \sin(-\frac{1}{2}\delta_{\mathfrak{w}} + \delta_{\mathfrak{w}}) = \sin(\frac{1}{2}\delta_{\mathfrak{w}}) = -\sin(\alpha_{lo})$. Hence

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$$\begin{aligned}
v'_{lo} =&(\mathfrak{v}(\sin(\alpha_{l_o}) + \sin(\alpha_{l_o} - \delta_{\mathfrak{v}})) + \mathfrak{w}(\sin(\alpha_{l_o}) + \sin(\alpha_{l_o} + \delta_{\mathfrak{w}})))\\
=&(\mathfrak{v}(\sin(\alpha_{l_o}) + \sin(\alpha_{l_o})\cos(\delta_{\mathfrak{v}}) - \cos(\alpha_{l_o})\sin(\delta_{\mathfrak{v}}))\\
&+ \mathfrak{w}(\sin(\alpha_{l_o}) + \sin(\alpha_{l_o})\cos(\delta_{\mathfrak{w}}) + \cos(\alpha_{l_o})\sin(\delta_{\mathfrak{w}})))\\
=&\mathfrak{v}\sin(\alpha_{l_o}) + \mathfrak{v}\sin(\alpha_{l_o})\cos(\delta_{\mathfrak{v}}) - \mathfrak{v}\cos(\alpha_{l_o})\sin(\delta_{\mathfrak{v}})\\
&+ \mathfrak{w}\sin(\alpha_{l_o}) + \mathfrak{w}\sin(\alpha_{l_o})\cos(\delta_{\mathfrak{w}}) + \mathfrak{w}\cos(\alpha_{l_o})\sin(\delta_{\mathfrak{w}})\\
=&\mathfrak{v}\sin(\alpha_{l_o}) + \mathfrak{v}\sin(\alpha_{l_o})\cos(\delta_{\mathfrak{v}}) + \mathfrak{w}\sin(\alpha_{l_o}) + \mathfrak{w}\sin(\alpha_{l_o})\cos(\delta_{\mathfrak{w}})\\
&- \mathfrak{v}\cos(\alpha_{l_o})\sin(\delta_{\mathfrak{v}}) + \mathfrak{w}\cos(\alpha_{l_o})\sin(\delta_{\mathfrak{w}})\\
=&\sin(\alpha_{l_o})(\mathfrak{v}(1 + \cos(\delta_{\mathfrak{v}})) + \mathfrak{w}(1 + \cos(\delta_{\mathfrak{w}}))) + \cos(\alpha_{l_o})(-\mathfrak{v}\sin(\delta_{\mathfrak{v}}) + \mathfrak{w}\sin(\delta_{\mathfrak{w}}))
\end{aligned}$$

Assuming that $-\mathfrak{w}\sin(\delta_{\mathfrak{w}})+\mathfrak{v}\sin(\delta_{\mathfrak{v}}) = 0$, then the eigenvalue requirement is fulfilled here with the same constant (λ) factor $(\mathfrak{v}(1+\cos(\delta_{\mathfrak{v}}))+\mathfrak{w}(1+\cos(\delta_{\mathfrak{w}})))$. Let us assume $\alpha_{lo} = -\frac{1}{2}\delta_{\mathfrak{w}} + \pi$ in which case $v_{lo} = \sin(\frac{1}{2}\delta_{\mathfrak{w}} + \pi) = -\sin(\frac{1}{2}\delta_{\mathfrak{v}})$. Then clearly $\sin(\alpha_{lo}+\delta_{\mathfrak{w}}) = \sin(-\frac{1}{2}\delta_{\mathfrak{w}}+\pi+\delta_{\mathfrak{w}}) = \sin(\frac{1}{2}\delta_{\mathfrak{w}}+\pi) = -sin(-\frac{1}{2}\delta_{\mathfrak{w}}+\pi) = -\sin(\alpha_{lo})$. By the same derivation as above, we can show that the eigenvalue requirement is satisfied.

Last even border node If it is even, let us take $\alpha_{le} = -\frac{1}{2}\delta_{\mathfrak{v}}$.

For an even node l_e, its eigenvector component amounts to $v_{le} = -(-1)^{l_e}\sin(\alpha_{le}) = -\sin(\alpha_{le})$, its preceding (odd) node component is $v_{l_e-1} = -(-1)^{l_e-1}\sin(\alpha_{le} - \delta_{\mathfrak{w}}) = \sin(\alpha_{le} - \delta_{\mathfrak{w}})$

Hence upon multiplication of the Laplacian matrix with the eigenvector the result for a last even l_e node would be

$$\begin{aligned}
v'_{le} =&\mathfrak{w}(-1)^{l_e-1}\sin(\alpha_{l_e} - \delta_{\mathfrak{w}}) - \mathfrak{w}(-1)^{l_e}\sin(\alpha_{l_e})\\
=&-\mathfrak{w}\sin(\alpha_{l_o} - \delta_{\mathfrak{w}}) - \mathfrak{w}\sin(\alpha_{l_o}) = -\mathfrak{w}(\sin(\alpha_{l_o} - \delta_{\mathfrak{w}}) + \sin(\alpha_{l_o}))
\end{aligned}$$

Let us assume $\alpha_{le} = -\frac{1}{2}\delta_{\mathfrak{v}}$ in which case $v_{le} = -\sin(\frac{1}{2}\delta_{\mathfrak{v}})$. Then clearly $\sin(\alpha_{le} + \delta_{\mathfrak{v}}) = \sin(-\frac{1}{2}\delta_{\mathfrak{v}} + \delta_{\mathfrak{v}}) = \sin(\frac{1}{2}\delta_{\mathfrak{v}}) = -\sin(\alpha_{le})$. Hence

$$\begin{aligned}
v'_{le} =&-\mathfrak{w}(\sin(\alpha_{l_e}) + \sin(\alpha_{l_e} - \delta_{\mathfrak{w}})) - \mathfrak{v}(\sin(\alpha_{l_e}) + \sin(\alpha_{l_e} + \delta_{\mathfrak{v}})))\\
=&-\sin(\alpha_{l_e})(\mathfrak{w}(1 + \cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1 + \cos(\delta_{\mathfrak{v}}))) - \cos(\alpha_{l_o})(-\mathfrak{w}\sin(\delta_{\mathfrak{w}}) + \mathfrak{v}\sin(\delta_{\mathfrak{v}}))
\end{aligned}$$

Assuming that $-\mathfrak{w}\sin(\delta_{\mathfrak{w}}) + \mathfrak{v}\sin(\delta_{\mathfrak{v}}) = 0$, then the eigenvalue requirement is fulfilled here with the same constant (λ) factor $(\mathfrak{v}(1+\cos(\delta_{\mathfrak{v}}))+\mathfrak{w}(1+\cos(\delta_{\mathfrak{w}})))$. Let us assume $\alpha_{le} = -\frac{1}{2}\delta_{\mathfrak{v}}+\pi$ in which case $v_{le} = -\sin(\frac{1}{2}\delta_{\mathfrak{v}}+\pi) = \sin(\frac{1}{2}\delta_{\mathfrak{v}})$. Then clearly $\sin(\alpha_{le} + \delta_{\mathfrak{v}}) = \sin(-\frac{1}{2}\delta_{\mathfrak{v}} + \pi + \delta_{\mathfrak{v}}) = \sin(\frac{1}{2}\delta_{\mathfrak{v}} + \pi) = -sin(-\frac{1}{2}\delta_{\mathfrak{v}} + \pi) = -\sin(\alpha_{le})$. By the same derivation as in preceding section, we find that the eigenvalue requirement is satisfied.

3.4 Materialization of Assumptions

We have made the following assumptions so far:

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- $-\mathfrak{v}\sin(\delta_{\mathfrak{v}})+\mathfrak{w}\sin(\delta_{\mathfrak{w}})=0$ (eq.(11)).
- Eigenvector elements for combinatorial Laplacian are of the form $v_{nodeid} = -(-1)^{nodeid}\sin(\alpha_{nodeid})$ (Sec. 3.3) or of the form $v_{nodeid}=\cos(\alpha_{nodeid})$ (Sec. 3.2)
- The first node angle is of the form $\alpha_1=\frac{1}{2}\delta_{\mathfrak{v}}$ (pages 9, 7), this fits eq.(9, 8, 6, 5)
- The above angles in the analytical form are related as follows: $\alpha_o=\alpha_{o-1}+\delta_{\mathfrak{v}}$ for odd nodes o and $\alpha_e=\alpha_{e-1}+\delta_{\mathfrak{v}}$ for even nodes e. This fits eq.(9, 8, 6, 5)
- If the last node is an odd node l_o, the angle of which is of the form $\alpha_{lo} = -\frac{1}{2}\delta_{\mathfrak{w}}$ or $\alpha_{lo}=-\frac{1}{2}\delta_{\mathfrak{w}}+\pi$ (page 8).
- If the last node is an even node l_e, its angle is of the form $\alpha_{le}=-\frac{1}{2}\delta_{\mathfrak{v}}$ or $\alpha_{le}=-\frac{1}{2}\delta_{\mathfrak{v}}+\pi$ (page 7)
- Eigenvalues for combinatorial Laplacian are of the form $\mathfrak{w}(1+\cos(\delta_{\mathfrak{w}}))+\mathfrak{v}(1+\cos(\delta_{\mathfrak{v}}))$ (page 8) or $\mathfrak{w}(1-\cos(\delta_{\mathfrak{w}}))+\mathfrak{v}(1-\cos(\delta_{\mathfrak{v}}))$, correspondingly. (page 6)

We need to check whether and when these assumptions, for which we do not know whether they fit the biweighted graph theorem, are true. Additionally, to ensure that we have an analytical form for the eigenvalues and eigenvectors, we need to ensure that we have as many orthogonal eigenvectors as there are nodes.

So let us check what the condition $-\mathfrak{v}\sin(\delta_{\mathfrak{v}})+\mathfrak{w}\sin(\delta_{\mathfrak{w}})=0$. implies for $\delta_{\mathfrak{w}}$ and $\delta_{\mathfrak{v}}$. Denote $\delta_{\mathfrak{wv}}=\delta_{\mathfrak{w}}+\delta_{\mathfrak{v}}$. Then

$$-\mathfrak{v}\sin(\delta_{\mathfrak{wv}}-\delta_{\mathfrak{w}})+\mathfrak{w}\sin(\delta_{\mathfrak{w}})=0$$
$$\tan(\delta_{\mathfrak{w}})=\frac{\mathfrak{v}\sin(\delta_{\mathfrak{wv}})}{\mathfrak{v}\cos(\delta_{\mathfrak{wv}})+\mathfrak{w}}$$
$$\delta_{\mathfrak{w}}=\arctan\frac{\mathfrak{v}\sin(\delta_{\mathfrak{wv}})}{\mathfrak{v}\cos(\delta_{\mathfrak{wv}})+\mathfrak{w}}$$

Similarly

$$\delta_{\mathfrak{v}}=\arctan\frac{\mathfrak{w}\sin(\delta_{\mathfrak{wv}})}{\mathfrak{w}\cos(\delta_{\mathfrak{wv}})+\mathfrak{v}}$$

This fits eq. (3) and (2) However, $\delta_{\mathfrak{v}},\delta_{\mathfrak{w}}$ shall not be computed simultaneously from the above formulas, but rather only one of them and the other from the sum $\delta_{\mathfrak{w}}\mathfrak{v}=\delta_{\mathfrak{w}}+\delta_{\mathfrak{v}}$ because of ambiguity of arctan in the range $[-\pi,\pi]$. If $\mathfrak{v}/\mathfrak{w}\geq 1$, then $\delta_{\mathfrak{v}}$ should be computed from the formula above, and otherwise $\delta_{\mathfrak{w}}$. The value of the computed δ should be taken from the range $(-\frac{\pi}{2},\frac{\pi}{2})$ so that the sine of that δ has he same sign as the sine of the right-hand side.

We have the requirement:

$$\nu_{[z],[2x]}=\cos(\frac{1}{2}\delta_{\mathfrak{v}}+x\delta_{\mathfrak{wv}}+\delta_{\mathfrak{w}})$$
$$\nu_{[z],[2x+1]}=-(-1)^{nodeid}\sin(\frac{1}{2}\delta_{\mathfrak{v}}+x\delta_{\mathfrak{wv}})$$
$$\nu_{[z],[2x]}=-(-1)^{nodeid}\sin(\frac{1}{2}\delta_{\mathfrak{v}}+(x-1)\delta_{\mathfrak{wv}}+\delta_{\mathfrak{w}})$$

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which needs to be aligned with the aforementioned restriction on the values of last noder angles. Note that we have introduced the restrictions for the last node: If it is even, then $\nu_{[z],[n]} = \cos(-\frac{1}{2}\delta_{\mathfrak{v}})$, and $\nu_{[z],[n]} = \sin(-\frac{1}{2}\delta_{\mathfrak{v}})$, respectively. Both may be true if $\frac{1}{2}\delta_{\mathfrak{v}} + (\frac{n}{2}-1)\delta_{\mathfrak{wv}} + \delta_{\mathfrak{w}} = -\frac{1}{2}\delta_{\mathfrak{v}} + k2\pi$ for some k. Equivalently $\delta_{\mathfrak{v}} + (\frac{n}{2}-1)\delta_{\mathfrak{wv}} + \delta_{\mathfrak{w}} = k2\pi$ or $\frac{n}{2}\delta_{\mathfrak{wv}} = k2\pi$ Also we have that if it is even, then $\nu_{[z],[n]} = \cos(-\frac{1}{2}\delta_{\mathfrak{v}}+\pi)$ or $\nu_{[z],[n]} = \sin(-\frac{1}{2}\delta_{\mathfrak{v}}+\pi)$ respectively. Both may be true if $\frac{1}{2}\delta_{\mathfrak{v}}+(\frac{n}{2}-1)\delta_{\mathfrak{wv}}+\delta_{\mathfrak{w}} = -\frac{1}{2}\delta_{\mathfrak{v}}+\pi+k2\pi$ for some k. That is $\delta_{\mathfrak{v}}+(\frac{n}{2}-1)\delta_{\mathfrak{wv}}+\delta_{\mathfrak{w}} = k2\pi+\pi$, so $\frac{n}{2}\delta_{\mathfrak{wv}} = k2\pi+\pi$. Summarizing both cases we get the criterion:

$$\frac{n}{2}\delta_{\mathfrak{wv}} = k'\pi$$

for some k'. Hence

$$\delta_{\mathfrak{wv}} = \frac{k'\pi}{0.5n}$$

as required by eq. (1).

If n is odd, then $\nu_{[z],[n]} = \cos(-\frac{1}{2}\delta_{\mathfrak{w}})$ or $\nu_{[z],[n]} = \sin(-\frac{1}{2}\delta_{\mathfrak{w}})$ or $\nu_{[z],[n]} = \cos(-\frac{1}{2}\delta_{\mathfrak{w}}+\pi)$ or $\nu_{[z],[n]} = \sin(-\frac{1}{2}\delta_{\mathfrak{w}}+\pi)$. The first two may be true if $\frac{1}{2}\delta_{\mathfrak{v}} + \frac{n-1}{2}\delta_{\mathfrak{wv}} = -\frac{1}{2}\delta_{\mathfrak{w}} + k2\pi$, so $\frac{1}{2}\delta_{\mathfrak{wv}} + \frac{n-1}{2}\delta_{\mathfrak{wv}} = k2\pi$, so $\frac{n}{2}\delta_{\mathfrak{wv}} = k2\pi$. The last two if $\frac{1}{2}\delta_{\mathfrak{wv}} + \frac{n-1}{2}\delta_{\mathfrak{wv}} = -\frac{1}{2}\delta_{\mathfrak{w}} + \pi + k2\pi$. That is $\frac{n}{2}\delta_{\mathfrak{wv}} = k2\pi+\pi$ So, as previously $\frac{n}{2}\delta_{\mathfrak{wv}} = k'\pi$ for some k'. Hence $\delta_{\mathfrak{wv}} = \frac{k'\pi}{0.5n}$ as required by eq. (1).

Note that in this way we get $2n$ eigenvectors and eigenvalues while only n eigenvalues/eigenvectors are possible. This means that there will be repetitions, which are easily detectable. In practice, the leading half of sine ad cosine related eigenvectors form the orthogonal basis plus the eigenvector related to the eigenvalue zero. More precisely: if n is even, then $1 : n/2$ first sine elements and $1 : n/2-1$ cosine elements plus nth element. If n is odd, then $1 : (n-1)/2$ first sine elements and $1 : (n-1)/2$ cosine elements plus nth element. If $z = n$, then cosine λ is equal zero and $\boldsymbol{\nu}$ is a constant vector. If n is even and $z = n/2$, then cosine $\boldsymbol{\nu}$ is a zero vector and hence respective λ should be ignored. The collapsing of eigenvalues can be explained as follows.

$$\begin{aligned}\lambda_{[z]} =& \mathfrak{w}(1-\cos(\delta_{\mathfrak{w}})) + \mathfrak{v}(1-\cos(\delta_{\mathfrak{v}}))\\ =& \mathfrak{w} + \mathfrak{v} - \sin(\delta_{\mathfrak{w}})(\mathfrak{w}\frac{\cos(\delta_{\mathfrak{w}})}{\sin(\delta_{\mathfrak{w}})} + \mathfrak{v}\frac{\cos(\delta_{\mathfrak{v}})}{\sin(\delta_{\mathfrak{w}})})\end{aligned}$$

Condition $-\mathfrak{v}\sin(\delta_{\mathfrak{v}}) + \mathfrak{w}\sin(\delta_{\mathfrak{w}}) = 0$ implies $\frac{\mathfrak{v}}{\sin(\delta_{\mathfrak{w}})} = \frac{\mathfrak{w}}{\sin(\delta_{\mathfrak{v}})}$. Therefore

$$\lambda_{[z]} = \mathfrak{w} + \mathfrak{v} - \mathfrak{w}\sin(\delta_{\mathfrak{w}})(\frac{\cos(\delta_{\mathfrak{w}})}{\sin(\delta_{\mathfrak{w}})} + \frac{\cos(\delta_{\mathfrak{v}})}{\sin(\delta_{\mathfrak{v}})})$$

Now recall that $\tan(\delta_{\mathfrak{w}}) = \frac{\mathfrak{v}\sin(\delta_{\mathfrak{wv}})}{\mathfrak{v}\cos(\delta_{\mathfrak{wv}})+\mathfrak{w}}$ and $\tan(\delta_{\mathfrak{v}}) = \frac{\mathfrak{w}\sin(\delta_{\mathfrak{wv}})}{\mathfrak{w}\cos(\delta_{\mathfrak{wv}})+\mathfrak{v}}$. Therefore we get

$$\lambda_{[z]} = \mathfrak{w} + \mathfrak{v} - \mathfrak{w}\sin(\delta_{\mathfrak{w}})\left(\frac{\mathfrak{v}\cos(\delta_{\mathfrak{wv}})+\mathfrak{w}}{\mathfrak{v}\sin(\delta_{\mathfrak{wv}})} + \frac{\mathfrak{w}\cos(\delta_{\mathfrak{wv}})+\mathfrak{v}}{\mathfrak{w}\sin(\delta_{\mathfrak{wv}})}\right)$$

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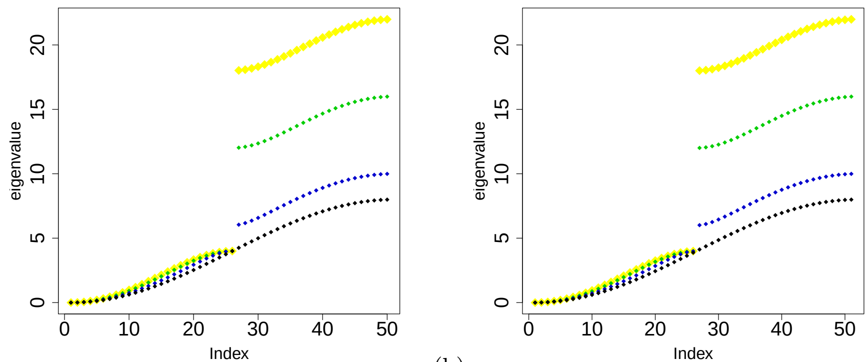

(a) (b)

Fig. 2. Eigenvalue spectrum for (a) $n = 50$ and (b) $n = 51$. $\mathfrak{w} = 2$, whereby $\mathfrak{v} = 9$ in yellow, $\mathfrak{v} = 6$ in green, $\mathfrak{v} = 3$ in blue, $\mathfrak{v} = 2$ in black.

$$\lambda_{[z]} = \mathfrak{w} + \mathfrak{v} - \mathrm{sign}(\sin(\delta_{\mathfrak{w}\mathfrak{v}}))\mathfrak{w}\sqrt{\frac{1}{1+\left(\frac{\cos(\delta_{\mathfrak{w}\mathfrak{v}})+\frac{\mathfrak{w}}{\mathfrak{v}}}{\sin(\delta_{\mathfrak{w}\mathfrak{v}})}\right)^2}\frac{2\cos(\delta_{\mathfrak{w}\mathfrak{v}})+\frac{\mathfrak{v}}{\mathfrak{w}}\frac{\mathfrak{w}}{\mathfrak{v}}}{\sin(\delta_{\mathfrak{w}\mathfrak{v}})}}$$

$$\lambda_{[z]} = \mathfrak{w} + \mathfrak{v} - \mathfrak{w}\sqrt{\frac{1}{1+(\frac{\cos(\delta_{\mathfrak{w}\mathfrak{v}})+\frac{\mathfrak{w}}{\mathfrak{v}}}{\sin(\delta_{\mathfrak{w}\mathfrak{v}})})^2}\frac{2\cos(\delta_{\mathfrak{w}\mathfrak{v}})+\frac{\mathfrak{v}}{\mathfrak{w}}\frac{\mathfrak{w}}{\mathfrak{v}}}{|\sin(\delta_{\mathfrak{w}\mathfrak{v}})|}}$$

Now consider $z' = n - z$. Then clearly the corresponding $\delta'_{\mathfrak{w}\mathfrak{v}} = 2\pi - \delta_{\mathfrak{w}\mathfrak{v}}$. As $\cos(2\pi - \alpha) = \cos(\alpha)$ and $|\sin(2\pi - \alpha)| = |\sin(\alpha)|$, we get that $\lambda_{[z']} = \lambda_{[z]}$. Similarly foir the sine based λs. Therefore we need to reject the eigenvalues and eigenvectors as described above. This completes the proof.

At the end, let us have a look at Figure 2, presenting eigenvalue spectrum for an even number of nodes $n = 50$ and for an odd number of nodes $n = 51$. It illustrates the changes to the spectrum when the proportion of alternating weights changes. $\mathfrak{w}$ was fixed at weights 2, while $\mathfrak{v}$ takes on values 9,6,3 and 2. One sees that if both weights are equal, the spectrum id somehow "continuous", while increasing disproportions move one part of the spectrum upwards.

3.5 Multidimensional Case

For multidimensional grids the eigenvalues are sums of component eigenvalues and the eigenvector components are products of component eigenvector components, like in case of weighted eigenvectors and eigenvalues.

Note that contrary to unweighted and weighted grids, the eigenvector components for biweighted grids depend on the weights.

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3.6 Eigensolutions of Unoriented Laplacians of Biweighted Grid Graphs

These are easily derived from the combinatorial Laplacian eigenvalues (identical) and eigenvectors (with alternating signs of components), just like in the unweighted and singly weighted case.

4 Conclusions

We have presented a closed-form method of computation of all eigenvalues and eigenvectors of a biweighted path grid graph for combinatorial Laplacians. Their properties may be of interest as generalisations of results of [5], [11], [12]. The closed-form formulas for eigenvalues and eigenvectors of bi-weighted grid graphs may be of high interest to researchers dealing with cluster analysis of graphs [8], especially with spectral cluster analysis, and compressive spectral clustering (CSC) [19]. While unweighted grid graphs can be considered as types of graphs that have no intrinsic cluster structure, the bi-weighted grid graphs can be considered as types of graphs that have either no intrinsic cluster structure (when the weights are equal) or the structure of which can be twisted in various ways. The weights permit to simulate node clusters not perfectly separated from each other, with various shades of this imperfection. This fact opens new possibilities for exploitation of closed-form form solutions eigenvectors and eigenvalues of graphs while testing and/or developing such algorithms and exploring their theoretical properties. This is particularly true for tests on grids with billions of nodes where typical numerical procedures suffer from space and time problems.

As increasing interest in weighted graph Laplacians exists, it would be an interesting research topic to find also closed form solutions to Laplacians of weighted graphs with other weighting schemas than those assumed in this work. Also the results presented here may be a starting point for finding solutions for normalized, random walk and other Laplacians in the spirit of the paper [12].

As mentioned in Sec.2, the biweighted grid graph problem may be viewed as a special case of tridiagonal matrices, investigated by [10], where in our case the sub- and superdiagonals are identical (though not constant). His theorems 3.2. and 3.4 are relevant here. Our results were derived independently of [10]. As one would expect, the derivation and the formulas are simpler, in particular for eigenvalues (e.g. no square rooting) and eigenvectors (e.g. either sine or cosine is computed once for each vector element). Our solution seems also be better suited for generalisations to normalised and random walk Laplacians because it follow the spirit of [12]. Normalised and random walk Laplacians cannot be handled by [10] as they are not tridiagonal matrices as defined in [10].

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