Multidimensional BSDEs With Mixed Reflections And Balance Sheet Optimal Switching Problem

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Abstract. In this paper, we study a system of multidimensional coupled reflected backward stochastic differential equations (RBSDEs) with interconnected generators and barriers and mixed reflections, i.e. oblique and normal reflections. This system of equations is arising in the context of optimal switching problem when both sides of the balance sheet are considered. This problem incorporates both the action of switching between investment modes and the action of abandoning the investment project before its maturity once it becomes unprofitable. Pricing such real options (switch option and abandon option) is equivalent to solve the system of coupled RBSDEs considered in the paper, for which we show the existence of a continuous adapted minimal solution via a Picard iteration method.

Keywords: Real options· Optimal switching· Balance sheet· Trade-off strategies· Merger and acquisition· Backward SDEs· Mixed reflections.

1 Introduction

Optimal switching problem (OSP) has attracted a lot of interest in the recent years (see among others [1,2,5,6,7,11,12]), since it can be related to many practical applications, for example the problem of valuation investment opportunities.

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OSP consists in finding an optimal management strategy for a production company that can run in m, $m \geq 2$, different modes . A management strategy δ is a combination of a nondecreasing sequence of stopping times $(\tau_n)_{n\geq 0}$, and a sequence of random variables $(\epsilon_n)_{n\geq 0}$ taking values in the set of possible production modes $\Lambda = \{1, \ldots, m\}$. At time τ_n, in order to maximize the profit of the company, the manager decides to switch the production from the current mode ϵ_{n-1} to ϵ_n. When the production of the company is working under a strategy δ, it generates a gain equal to $J(\delta)$. The OSP amounts to finding an optimal management strategy δ^* such that $J(\delta^*) = \sup_{\delta} J(\delta)$. The OSP is connected with multidimensional RBSDEs with oblique reflections and interconnected barriers.

One dimensional BSDEs with normal reflections were first introduced by [10]. The multidimensional case was studied by Gegout-Petit and Pardoux [9], and then further investigated in many other works see e.g. [8,13]. Multidimensional BSDEs with oblique reflections occurring in the context of OSPs were first introduced by [12]. They consider RBSDEs with generator taking the form $f_i(\cdot, y^i_\cdot, z^i_\cdot)$ and barrier $\min_{j\in\Lambda^{-i}} \left(y^j_\cdot + g_{i,j}\right)$ where $g_{i,j}$ are constant switching costs and $\Lambda^{-i} = \Lambda - \{i\}$. Later, Hamadène and Zhang [11] generalized the preceding work by considering general generators and barriers of the following types $f_i(\cdot, y^1_\cdot, \ldots, y^m_\cdot, z^i_\cdot)$ and $\max_{j\in\Lambda^{-i}} h_{i,j}(., y^j_\cdot)$. Xu [17] dealt with the same kind of RBSDEs but when the generator, which is discontinuous w.r.t. y^i, and the barrier take respectively the following forms $f_i(\cdot, y^i_\cdot, z^i_\cdot)$ and $\max_{j\in\Lambda^{-i}} \left(y^j_\cdot - g_{i,j}\right) \vee S^i_\cdot$. Then, Aazizi et al. [1] extended the results of [17] to the case of generators and barriers of the form $f_i(\cdot, y^1_\cdot, \ldots, y^m_\cdot, z^i_\cdot)$ and $\max_{j\in\Lambda^{-i}} h_{i,j}(., y^j_\cdot) \vee S^i_\cdot$.

In this paper, we are interested by Balance sheet OSP (BSOSP) which is a combination between the classical OSP described above and optimal stopping involving the balance sheet. BSOSP incorporates both the action of switching between modes and the action of abandoning a project once it becomes unprofitable. There are only few papers dealing with BSOSPs. Djehiche and Hamdi [4] considered the 2-modes case, i.e. $\Lambda = \{1, 2\}$. Their generators are of the form $f^+_i(\cdot, Y^{+,i}_\cdot, Z^{+,i}_\cdot)$, $f^-_i(\cdot, Y^{-,i}_\cdot, Z^{-,i}_\cdot)$ and their barriers of type $(Y^{+,j}_\cdot - g_{i,j}(.)) \vee (Y^{-,i}_\cdot - C_i(.))$ and $(Y^{-,j}_\cdot + g_{i,j}(.)) \vee (Y^{+,i}_\cdot + B_i(.))$, where C_i and B_i are switching costs and $j \in \Lambda^{-i}$. Recently, the BSOSP multi-modes case was solved by Eddahbi et al. [5] when the barriers are of the form $\max_{j\in\Lambda^{-i}} (Y^{+,j}_\cdot - g_{i,j}(.)) \vee (Y^{-,i}_\cdot + C^i(.))$ and $Y^{+,i}_\cdot + B_i(.)$ (see Eddahbi et al. [6] for the mean–field case).

Now, let us describe precisely the problem studied in this paper by introducing some notations. Let $T > 0$ be a given real number, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space endowed with a d–dimensional Brownian motion $W = (W_t)_{0\leq t\leq T}$. $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is the natural filtration of the Brownian motion augmented by the $\mathbb{P}$–null sets of $\mathcal{F}$. All the measurability notion will refer to this filtration. The euclidean norm of a vector $z \in I\!R^d$ is denoted $|z|$. Furthermore, we introduce the following spaces of processes. $\mathbb{L}^2$ is the space of $\mathbb{R}$–valued processes ξ, such that $||\xi||_{\mathbb{L}^2} := (E\left[|\xi|^2\right])^{1/2} < +\infty$. $\mathcal{S}^2$ (resp.

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$\mathcal{S}_c^2$) is the set of $\mathbb{R}$–valued adapted and continuous (resp. càdlàg) processes $(Y_t)_{0\leq t\leq T}$ such that $||Y||_{\mathcal{S}^2}$ (resp. $||Y||_{\mathcal{S}_c^2}$) $:= \left(E\left[\sup_{0\leq t\leq T}|Y_t|^2\right]\right)^{1/2} < +\infty$. $\mathcal{M}^{d,2}$ is the set of $\mathbb{R}^d$–valued, progressively measurable processes $(Z_t)_{0\leq t\leq T}$ such that $||Z||_{\mathcal{M}^{d,2}} := \left(E\left[\int_0^T|Z_s|^2ds\right]\right)^{1/2} < +\infty$. $\mathcal{K}^2$ (resp. $\mathcal{K}_c^2$) is the set of non-decreasing processes K, satisfying $K_0 = 0$ and that belong to $\mathcal{S}^2$ (resp. $\mathcal{S}_c^2$).

Next, to illustrate the BSOSP studied in this paper, let us deal with a concrete example. Consider a company that has m modes of production (if $m = 3$, minimal, average and maximal production modes). The manager of the company has two options. A switch option, i.e. in order to maximize its global profit, she switches the production between the modes depending on their random performances but this switching incorporates a cost called switching cost. The manager has also an abandon option i.e. stop the production once it becomes unprofitable. More precisely, being in mode $i \in \Lambda$, one have to switch at time t to another mode $j \in \Lambda^{-i}$, once we have that the expected profit $Y^{+,i}$ in this mode falls below the following barrier

$$Y_t^{+,i} \leq S_t^{+,i} := \max_{j\in\Lambda^{-i}} h_{i,j}(t, Y_t^{+,j}) \vee (Y_t^{-,i} + C^i(t)), \forall\, t \in [0, T], \qquad (1)$$

where $h_{i,j}$ is nonlinear random function (a special case is when $h_{i,j}(.,y) = y - g_{i,j}$, where $g_{i,j}$ is a switching cost from mode i to mode j), $Y^{-,i}$ is the expected cost in mode i, and C^i is the cost incurred when exiting/terminating the production while in mode i. Since we consider both sides of the balance sheet, the manager has to switch at time t to another mode $j \in \Lambda^{-i}$, as soon as the expected cost in mode i, $Y^{-,i}$ rises above the following barrier

$$Y_t^{-,i} \geq S_t^{-,i} := S_t^i \wedge \left(Y_t^{+,i} + B^i(t)\right), \forall\, t \in [0, T], \qquad (2)$$

where $S^i(t)$ is a cost of default (i.e. in this case the project is no longer profitable and thus leads to the abandon of this latter even before its maturity), and B^i is the benefit incurred when exiting/terminating the production while in mode i. It is well known that the BSOSP can be formulated using the following system of Snell envelopes

$$Y_t^{+,i} = ess\sup_{\tau\geq t} E\left[\int_t^\tau f_i^+(s)ds + S_\tau^{+,i}\mathbf{1}_{[\tau<T]} + \xi_i^+\mathbf{1}_{[\tau=T]}|\mathcal{F}_t\right], \qquad (3)$$

$$Y_t^{-,i} = ess\inf_{\tau\geq t} E\left[\int_t^\tau f_i^-(s)ds + S_\tau^{-,i}\mathbf{1}_{[\tau<T]} + \xi_i^-\mathbf{1}_{[\tau=T]}|\mathcal{F}_t\right], \qquad (4)$$

where $\tau \in [0, T]$ are $\mathbb{F}$–stopping times which represent the exit times from the production in mode i, f_i^+ and f_i^- denote respectively the running profit and cost per unit time dt and ξ_i^+ and ξ_i^- are respectively the values at time T of the profit and the cost yields.

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The BSOSP consists in showing existence and uniqueness of the processes $(Y^{+,i}, Y^{-,i})_{i\in\Lambda}$ and also proving that the following stopping times are optimal

$$\tau^{+,i} = \inf\left\{s \geq t : Y_s^{+,i} = S_s^{+,i}\right\} \wedge T, \text{and } \tau^{-,i} = \inf\left\{s \geq t : Y_s^{-,i} = S_s^{-,i}\right\} \wedge T.$$

Since the Snell envelope is strongly connected to RBSDEs, solving the BSOSP is equivalent to showing existence of continuous solution to the following general (sine we take $f_i^+(.) = f_i^+(s, \overrightarrow{Y}_s^+, Z_s^{+,i})$ $f_i^-(s) = f_i^-(s, \overrightarrow{Y}_s^-, Z_s^{-,i})$ where $\overrightarrow{Y}^+ := (Y^{+,1}, \ldots, Y^{+,m})$, $\overrightarrow{Y}^- := (Y^{-,1}, \ldots, Y^{-,m})$) system of BSDEs with mixed reflections: for $i \in \Lambda := \{1, \ldots, m\}$

$$(S)\begin{cases} Y_t^{+,i} = \xi_i^+ + \displaystyle\int_t^T f_i^+(s, \overrightarrow{Y}_s^+, Z_s^{+,i})ds - \int_t^T Z_s^{+,i} dW_s + K_T^{+,i} - K_t^{+,i}, \\ Y_t^{+,i} \geq S_t^{+,i}, \text{ and} \displaystyle\int_0^T \left[Y_s^{+,i} - S_s^{+,i}\right] dK_s^{+,i} = 0, \qquad (5) \\ Y_t^{-,i} = \xi_i^- + \displaystyle\int_t^T f_i^-(s, \overrightarrow{Y}_s^-, Z_s^{-,i})ds - \int_t^T Z_s^{-,i} dW_s - K_T^{-,i} + K_t^{-,i}, \\ Y_t^{-,i} \leq S_t^{-,i}, \text{ and } \displaystyle\int_0^T \left[S_s^{-,i} - Y_s^{-,i}\right] dK_s^{-,i} = 0, \qquad (6) \end{cases}$$

where T is called the time horizon, ξ_i^+ and ξ_i^- are called the terminal conditions, the random functions $f_i^+(\omega, t, \overrightarrow{y}, z^i) : \Omega \times [0, T] \times I\!R^m \times I\!R^d \to I\!R$ and $f_i^-(\omega, t, \overrightarrow{y}, z^i) : \Omega \times [0, T] \times I\!R^m \times I\!R^d \to I\!R$ are respectively $\mathcal{F}_t$–progressively measurable for each $(\overrightarrow{y}, z^i)$, called the generators. $h_{i,j}$ is a real nonlinear random function, and $C^i := (C^i(t))_{t\in[0,T]}$, $B^i := (B^i(t))_{t\in[0,T]}$, and $S^i := (S^i(t))_{t\in[0,T]}$ are previously given $(\mathcal{F}_t)_{0\leq t\leq T}$–adapted processes with some suitable regularity. The unknowns are the processes $(Y^{\pm,i}, Z^{\pm,i}, K^{\pm,i}) := (Y_t^{\pm,i}, Z_t^{\pm,i}, K_t^{\pm,i})_{t\in[0,T]}$ which are required to be $(\mathcal{F}_t)_{0\leq t\leq T}$–adapted. Moreover, $K^{+,i}$ and $K^{-,i}$ are non-decreasing processes. The second condition in (5) (resp. (6)) says that the first component $Y^{+,i}$ (resp. $Y^{-,i}$) of the solution of RBSDE (5) (resp. (6)) is forced to stay above (resp. below) the barrier $S_\cdot^{+,i}$ (resp. $S_\cdot^{-,i}$). The role of $K^{+,i}$ (resp. $K^{-,i}$) is to push $Y^{+,i}$ (resp. $Y^{-,i}$) upwards (resp. downwards) in order to keep it above (resp. below) the respective barrier in a minimal way in the sense of the third condition of RBSDE (5) (resp. (6)) which is called the minimal boundary condition i.e. $K^{+,i}$ (resp. $K^{-,i}$) increases only when $Y^{+,i}$ (resp. $Y^{-,i}$) touches the respective barrier.
Let us make precise the notion of a solution of the system of RBSDEs (S).

Definition 1. *A 6-uplet of processes* $(Y^{+,i}, Z^{+,i}, K^{+,i}, Y^{-,i}, Z^{-,i}, K^{-,i})$ *is called solution of the system of RBSDEs (S) if the two triples* $(Y^{+,i}, Z^{+,i}, K^{+,i})$ *and* $(Y^{-,i}, Z^{-,i}, K^{-,i})$ *belong to* $\mathcal{S}^2 \times \mathcal{M}^{d,2} \times \mathcal{K}^2$ *and satisfy the system (S).*

The main contribution of our paper is to establish the existence of a continuous minimal adapted solution to system of RBSDEs (S). To this end we use a Picard iteration method (see El Karoui et al. [10] for more details). Uniqueness of the solution does not hold, since it is not verified even for the two-modes

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case and for a less general form of RBSDE (S) (see the counter-example in [4, subsection 3.1]).

Clearly, our results generalize the related works in the literature, since our RBSDE (S) is more general in many features. Actually, the expected profits and cost yields $\overrightarrow{Y}^+$and $\overrightarrow{Y^-}$are respectively interconnected in the generators f_i^+ and f_i^-. This dependence can be interpreted as a nonzero-sum game problem, where the players' utilities affect each other. Furthermore, the solutions $Y^{+,i}$ and $Y^{-,i}$ are also interconnected in the barriers. Note that, the general barrier $h_{i,j}(\cdot, y)$ which is random and nonlinear, makes the dependence on the unknown process implicit. This, allows one to consider more general switching cost, for instance in the case of risk sensitive switching problem.

The remainder of the paper is organized as follows. Section 2 is devoted to the assumptions. In Section 3, we state and prove the main result of the paper.

2 Assumptions

Let us introduce the following assumptions:
[H1]: For each $i \in \Lambda$, f_i^+ and f_i^- satisfies:
(i) $\mathbb{E}\bigg(\int_0^T \sup_{\overrightarrow{y}: y_i=0} \left| f_i^+(t, \overrightarrow{y}, 0) \right|^2 dt + \int_0^T \sup_{\overrightarrow{y}: y_i=0} \left| f_i^-(t, \overrightarrow{y}, 0) \right|^2 dt \bigg) < +\infty.$
(ii) The mappings $(t, \overrightarrow{y}, z^i) \to f_i^+(t, \overrightarrow{y}, z^i)$ and $(t, \overrightarrow{y}, z^i) \to f_i^-(t, \overrightarrow{y}, z^i)$ are Lipschitz continuous in (y^i, z^i) uniformly in t, and are continuous in y^j for $j \in \Lambda^{-i}$.
(iii) $f_i^+(t, \overrightarrow{y}, z^i)$ and $f_i^-(t, \overrightarrow{y}, z^i)$ are increasing in y^j for $j \in \Lambda^{-i}$. This assumption means that the m-players are partners.
[H2]: For each $i, j \in \Lambda$, $h_{i,j}$ satisfies:
(i) $h_{i,j}(t, y)$ is continuous in (t, y);
(ii) $h_{i,j}(t, y)$ is increasing in y;
(iii) $h_{i,j}(t, y) \leq y$.
(iv) There is no sequence $i_2 \in \Lambda^{-i_1}, \ldots, i_k \in \Lambda^{-i_{k-1}}$, $i_1 \in \Lambda^{-i_k}$, and $(y^1, \ldots, y^k)$ such that $y^1 = h_{i_1,i_2}(t, y^2)$, $y^2 = h_{i_2,i_3}(t, y^3), \ldots, y_{k-1} = h_{i_{k-1},i_k}(t, y^k)$, $y^k = h_{i_k,i_1}(t, y^1)$. This means that there is no free loop of instantaneous switchings.

[H3]: For $t \in [0, T]$ and $i \in \Lambda$, $B^i(t)$, $C^i(t)$ and $S^i(t)$ belong to $\mathcal{S}^2$.
[H4]: For any $i \in \Lambda$ the random variables ξ_i^+ and ξ_i^- are $\mathcal{F}_T$–measurable and belong to $\mathbb{L}^2$. Moreover we assume that

$$\xi_i^+ \geq \max_{j \in \Lambda^{-i}} h_{i,j}(T, \xi_j^+) \vee (\xi_i^- + C^i(T)), \quad \text{and} \quad \xi_i^- \leq S^i(T) \wedge \big(\xi_i^+ + B^i(T)\big).$$

[H5]: For every $i \in \Lambda$, the processes $(B^i(t))_{0 \leq t \leq T}$ and $(S^i(t))_{0 \leq t \leq T}$ are semimartingales of the form $B^i(t) = B^i(0) + \int_0^t U^i(s) ds + \int_0^t V^i(s) dW_s$ and $S^i(t) = S^i(0) + \int_0^t \bar{U}^i(s) ds + \int_0^t \bar{V}^i(s) dW_s$ where $(U^i(t), \bar{U}^i(t))$ and $(V^i(t), \bar{V}^i(t))$ are respectively $(\mathbb{R})^2$ and $(\mathbb{R}^d)^2$-valued $\mathcal{F}_t$–progressively measurable processes which are $dt \otimes d\mathbb{P}$–square integrable.

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3 Main result

Next we state and prove the main result of this paper.

Theorem 1. *Assume that* **[H1]**–**[H5]** *hold. Then for all* $i \in \Lambda$*, the system of RBSDEs (S) admits a continuous minimal solution* $(Y^{\pm,i}, Z^{\pm,i}, K^{\pm,i})$.

Proof. The whole proof is performed in six steps.
Step 1: Construction of Picard's sequence of solutions
Consider the following sequence of RBSDEs defined recursively, for $i \in \Lambda$ and $t \in [0,T]$, as follows: For $n = 0$ we start with the following BSDE:

$$Y_t^{+,i,0} = \xi_i^+ + \int_t^T \underline{f}_i^+(s, Y_s^{+,i,0}, Z_s^{+,i,0})ds - \int_t^T Z_s^{+,i,0} dW_s, \tag{7}$$

and RBSDE:

$$\begin{cases} Y_t^{-,i,0} = \xi_i^- + \displaystyle\int_t^T \underline{f}_i^-(s, Y_s^{-,i,0}, Z_s^{-,i,0})ds - \int_t^T Z_s^{-,i,0} dW_s - K_T^{-,i,0} + K_t^{-,i,0}, \\ Y_t^{-,i,0} \leq S^i(t) \wedge \big(Y_t^{+,i,0} + B^i(t)\big), \\ 0 = \displaystyle\int_0^T \left[S^i(t) \wedge \big(Y_t^{+,i,0} + B^i(t)\big) - Y_t^{-,i,0}\right] dK_t^{-,i,0}, \end{cases} \tag{8}$$

where $\underline{f}_i^+(s,y,z^i) = \inf_{\overrightarrow{y}: y_i = y} f_i^+(s, \overrightarrow{y}, z^i)$ and $\underline{f}_i^-(s,y,z^i) = \inf_{\overrightarrow{y}: y_i = y} f_i^-(s, \overrightarrow{y}, z^i)$.
Now, for $n = 0$ consider the following system of RBSDEs:

$$\begin{cases} Y_t^{-,i,1} = \xi_i^- - \displaystyle\int_t^T Z_s^{-,i,1} dW_s - K_T^{-,i,1} + K_t^{-,i,1} \\ + \displaystyle\int_t^T f_i^-(s, Y_s^{-,1,0}, \ldots, Y_s^{-,i-1,0}, Y_s^{-,i,1}, Y_s^{-,i+1,0}, \ldots, Y_s^{-,m,0}, Z_s^{-,i,1})ds, \\ Y_t^{-,i,1} \leq S^i(t) \wedge \big(Y_t^{+,i,0} + B^i(t)\big), \\ 0 = \displaystyle\int_0^T \left[S^i(t) \wedge \big(Y_t^{+,i,0} + B^i(t)\big) - Y_t^{-,i,1}\right] dK_t^{-,i,1}, \\ Y_t^{+,i,1} = \xi_i^+ - \displaystyle\int_t^T Z_s^{+,i,1} dW_s + K_T^{+,i,1} - K_t^{+,i,1} \\ + \displaystyle\int_t^T f_i^+(s, Y_s^{+,1,0}, \ldots, Y_s^{+,i-1,0}, Y_s^{+,i,1}, Y_s^{+,i+1,0}, \ldots, Y_s^{+,m,0}, Z_s^{+,i,1})\, ds, \\ Y_t^{+,i,1} \geq \max_{j \in \Lambda^{-i}} h_{i,j}(t, Y_t^{+,j,0}) \vee \Big(Y_t^{-,i,1} + C^i(t)\Big), \\ 0 = \displaystyle\int_0^T \left[Y_t^{+,i,1} - \max_{j \in \Lambda^{-i}} h_{i,j}(t, Y_t^{+,j,0}) \vee \Big(Y_t^{-,i,1} + C^i(t)\Big)\right] dK_t^{+,i,1}. \end{cases} \tag{9}$$

Note that by **[H1]**(i) and (ii) we have that $\underline{f}_i^+$ and $\underline{f}_i^-$ are uniformly Lipschitz continuous in (y, z^i) and satisfy the following integrability condition

$$\mathbb{E}\Big\{ \int_0^T \Big(|\underline{f}_i^+(t,0,0)|^2 + |\underline{f}_i^-(t,0,0)|^2\Big)\, dt \Big\} < +\infty.$$

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Thus, from [14] it follows that for each $i \in \Lambda$ BSDE (7) admits a unique solution $(Y^{+,i,0}, Z^{+,i,0}) \in \mathcal{S}^2 \times \mathcal{M}^{d,2}$. Thus, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\left(S^i(t) \wedge \left(Y_t^{+,i,0} + B^i(t)\right)\right)^+\right|^2\right] \leq C < +\infty, \tag{10}$$

and thus in view of [10, Proposition 2.3] we deduce that RBSDE (8) has a unique solution $(Y^{-,i,0}, Z^{-,i,0}, K^{-,i,0}) \in \mathcal{S}^2 \times \mathcal{M}^{d,2} \times \mathcal{K}^2$.
As a by product, under the assumptions **[H1]**–**[H4]**, in view of [10, Proposition 2.3] the solution $(Y^{-,i,1}, Z^{-,i,1}, K^{-,i,1}) \in \mathcal{S}^2 \times \mathcal{M}^{d,2} \times \mathcal{K}^2$ exists and is unique. This in turn, in view of the following estimate, which holds due to assumption **[H2]**(iii), **[H3]** and the fact that $Y^{-,i,1} \in \mathcal{S}^2$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\left(\max_{j\in\Lambda^{-i}} h_{i,j}(t, Y_t^{+,j,0}) \vee \left(Y_t^{-,i,1} + C^i(t)\right)\right)^+\right|^2\right] \leq C < +\infty,$$

combined with [10, Proposition 2.3], leads to the existence of the unique solution $(Y^{+,i,1}, Z^{+,i,1}, K^{+,i,1}) \in \mathcal{S}^2 \times \mathcal{M}^{d,2} \times \mathcal{K}^2$.
Next, for $n \geq 1$, consider the following system

$$\begin{cases} Y_t^{-,i,n+1} = \xi_i^- + \displaystyle\int_t^T f_i^-(s, Y_s^{-,1,n}, \ldots, Y_s^{-,i-1,n}, Y_s^{-,i,n+1}, Y_s^{-,i+1,n}, \ldots \\ \qquad \ldots, Y_s^{-,m,n}, Z_s^{-,i,n+1})ds - \displaystyle\int_t^T Z_s^{-,i,n+1} dW_s - K_T^{-,i,n+1} + K_t^{-,i,n+1}, \\ Y_t^{-,i,n+1} \leq S^i(t) \wedge \left(Y_t^{+,i,n} + B^i(t)\right), \\ 0 = \displaystyle\int_0^T \left[S^i(t) \wedge \left(Y_t^{+,i,n} + B^i(t)\right) - Y_t^{-,i,n+1}\right] dK_t^{-,i,n+1}, \\ Y_t^{+,i,n+1} = \xi_i^+ + \displaystyle\int_t^T f_i^+(s, Y_s^{+,1,n}, \ldots, Y_s^{+,i-1,n}, Y_s^{+,i,n+1}, Y_s^{+,i+1,n}, \ldots \\ \qquad \ldots, Y_s^{+,m,n}, Z_s^{+,i,n+1})ds - \displaystyle\int_t^T Z_s^{+,i,n+1} dW_s + K_T^{+,i,n+1} - K_t^{+,i,n+1}, \\ Y_t^{+,i,n+1} \geq \max\limits_{j\in\Lambda^{-i}} h_{i,j}(t, Y_t^{+,j,n}) \vee (Y_t^{-,i,n+1} + C^i(t)), \\ 0 = \displaystyle\int_0^T \left[Y_t^{+,i,n+1} - \max_{j\in\Lambda^{-i}} h_{i,j}(t, Y_t^{+,j,n}) \vee \left(Y_t^{-,i,n+1} + C^i(t)\right)\right] dK_t^{+,i,n+1}. \end{cases} \tag{11}$$

Based on the arguments used previously, we can show by using an induction argument that for any $n \geq 2$, the system of RBSDEs (11) has a unique solution $(Y^{+,i,n}, Z^{+,i,n}, K^{+,i,n}, Y^{-,i,n}, Z^{-,i,n}, K^{-,i,n}) \in (\mathcal{S}^2)^2 \times (\mathcal{M}^{d,2})^2 \times (\mathcal{K}^2)^2,\ \forall i \in \Lambda$.

Step 2: Convergence of the sequences $(Y^{\pm,i,n})_{n\geq 0}$
Let us set,

$$\widehat{f}_i^-(s, y, z^i) = \sup_{\overrightarrow{y}: y_i = y} f_i^-(s, \overrightarrow{y}, z^i) \quad \text{and} \quad \dot{f}_i^+(s, y, z^i) = \sup_{\overrightarrow{y}: y_i = y} f_i^+(s, \overrightarrow{y}, z^i).$$

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Note that by **[H1]**(i) and (ii) we have that $\widehat{f}_i^-$ and $\dot{f}_i^+$ are uniformly Lipschitz continuous in (y, z^i) and satisfy the following integrability condition

$$\mathbb{E}\bigg\{\int_0^T \Big(|\widehat{f}_i^-(t,0,0)|^2 + |\dot{f}_i^+(t,0,0)|^2\Big)\, dt\bigg\} < +\infty. \tag{12}$$

Consider the following BSDE

$$\widehat{Y}_t = \sum_{i=1}^m |\xi_i^-| + \int_t^T \sum_{i=1}^m |\widehat{f}_i^-(s,\widehat{Y}_s,\widehat{Z}_s^i)|ds - \int_t^T \widehat{Z}_s dW_s.$$

It follows from [14] that this BSDE admits a unique solution $(\widehat{Y}_t,\widehat{Z}_t)\in\mathcal{S}^2\times\mathcal{M}^{d,2}$. Next, let $(\dot{Y}^i,\dot{Z}^i,\dot{K}^i)$ be solutions of the following system of reflected BSDEs, for any $i\in\Lambda$ and $t\in[0,T]$, as follows

$$\begin{cases} \dot{Y}_t^i = \sum_{i=1}^m |\xi_i^+| + \sum_{i=1}^m |\xi_i^-| + |C^i(T)| + \int_t^T \sum_{i=1}^m |\dot{f}_i^+(s,\widehat{Y}_s,\widehat{Z}_s^i)|ds \\ \qquad - \int_t^T \dot{Z}_s dW_s + \dot{K}_T^i - \dot{K}_t^i, \\ \dot{Y}_t^i \geq \max_{j\in\Lambda^{-i}} h_{i,j}(t,\dot{Y}_t^j) \vee \big(\widehat{Y}_t + C^i(t)\big), \\ \int_0^T \Big[\dot{Y}_s^i - \max_{j\in\Lambda^{-i}} h_{i,j}(s,\dot{Y}_s^j) \vee \big(\widehat{Y}_s + C^i(s)\big)\Big]\, d\dot{K}_s^i = 0. \end{cases} \tag{13}$$

By using previous arguments, and thanks to the fact that $\widehat{Y}_t\in\mathcal{S}^2$ and assumption **[H3]**, applying [1, Theorem 3.1] yields that this RBSDE admits a solution $(\dot{Y}^i,\dot{Z}^i,\dot{K}^i)\in\mathcal{S}^2\times\mathcal{M}^{d,2}\times\mathcal{K}^2$. Moreover, the following holds Next, by using an induction argument, plus a repeated use of the comparison theorem, we can easily show that for any $i\in\Lambda$, $t\in[0,T]$, $\forall n$:

$$Y_t^{-,i,0} \leq Y_t^{-,i,n} \leq Y_t^{-,i,n+1} \leq \widehat{Y}_t \quad \text{and} \quad Y_t^{+,i,0} \leq Y_t^{+,i,n} \leq Y_t^{+,i,n+1} \leq \dot{Y}_t^i, \ \text{a.s.} \tag{14}$$

Consequently, we deduce the following

$$\sup_{n\geq1}\mathbb{E}[\sup_{t\in[0,T]}|Y_t^{+,i,n}|^2] \leq \mathbb{E}[\sup_{t\in[0,T]}|Y_t^{+,i,0}|^2] + \mathbb{E}[\sup_{t\in[0,T]}|\dot{Y}_t^i|^2] < \infty, \ \forall i\in\Lambda, \tag{15}$$

$$\sup_{n\geq1}\mathbb{E}[\sup_{t\in[0,T]}|Y_t^{-,i,n}|^2] \leq \mathbb{E}[\sup_{t\in[0,T]}|Y_t^{-,i,0}|^2] + \mathbb{E}[\sup_{t\in[0,T]}|\widehat{Y}_t|^2] < \infty, \ \forall i\in\Lambda. \tag{16}$$

Next, from (14) combined with (15) and (16) we deduce that the sequences $\{Y^{+,i,n}\}_{n\geq0}$ and $\{Y^{-,i,n}\}_{n\geq0}$ admit limits. Therefore, let $Y^{+,i}$ and $Y^{-,i}$, $i\in\Lambda$ be two optional processes which are respectively the limits of $Y^{+,i,n}$ and $Y^{-,i,n}$. Applying Fatou's Lemma and the dominated convergence theorem, we obtain

$$\mathbb{E}[\sup_{t\in[0,T]}|Y_t^{\pm,i}|^2] < \infty, \quad \lim_{n\to\infty}\mathbb{E}\int_0^T |Y_t^{\pm,i,n} - Y_t^{\pm,i}|^2\, dt = 0, \ i\in\Lambda. \tag{17}$$

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Step 3: Uniform estimates for the sequences $\{(Z^{\pm,i,n},K^{\pm,i,n})\}_{n\geq 0}$, $i\in\Lambda$
By **[H2]**(iii), we obtain in view of the facts $(\widehat{Y}_t,\widehat{Z}_t)\in\mathcal{S}^2\times\mathcal{M}^{d,2}$, $(\dot{Y}^i,\dot{Z}^i,\dot{K}^i)\in\mathcal{S}^2\times\mathcal{M}^{d,2}\times\mathcal{K}^2$ and (14) combined with **[H3]** the following estimate of the barriers of RBSDE (11): for all $n\geq 1$ and all $i\in\Lambda$

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\left(\max_{j\in\Lambda^{-i}} h_{i,j}(t,Y_t^{+,j,n})\vee(Y_t^{-,i,n+1}+C^i(t))\right)^+\right|^2\right]<+\infty, \quad (18)$$

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\left(S^i(t)\wedge\left(Y_t^{+,i,n}+B^i(t)\right)\right)^+\right|^2\right]<+\infty. \quad (19)$$

Finally, with the estimates (15), (16), (18) and (19) at hand, applying the results in [10] we obtain that

$$\sup_{n\geq 1}\mathbb{E}[\int_0^T|Z_t^{\pm,i,n}|^2\,dt]<\infty,\quad \sup_{n\geq 1}\mathbb{E}|K_T^{\pm,i,n}|^2<\infty,\ i\in\Lambda. \quad (20)$$

Step 4: Continuity of the limit processes $Y^{-,i}$ and $Y^{+,i}$, $i\in\Lambda$.
To this end, let us first establish the absolute continuity of the increasing process $K^{-,i,n}$ w.r.t t for every $n\geq 0$.
We will first show that the claim holds true for $n=0$. Let

$$\Xi_t^i:=S_t^i\wedge\left(Y_t^{+,i,0}+B^i(t)\right)=Y_t^{+,i,0}+B^i(t)-\left(Y_t^{+,i,0}+B^i(t)-S_t^i\right)^+.$$

Applying Itô–Tanaka formula to Ξ_t^i, and in view of assumption [**H5**] we obtain

$$\Xi_t^i=\Xi_0^i+\int_0^t M_s^i ds+\int_0^t N_s^i dW_s-\frac{1}{2}L_t^i, \quad (21)$$

where $\{L_t^i, 0\leq t\leq T\}$ is the local time at 0 of the continuous semimartingale $\{Y_t^{+,i,0}+B^i(t)-S^i(t)\}$,

$$M_t^i:=-\mathbf{1}_{\{Y_t^{+,i,0}+B^i(t)>S^i(t)\}}\Big(-\underline{f}_i^+(s,Y_s^{+,i,0},Z_s^{+,i,0})+U_t^i-\bar{U}_t^i\Big)$$
$$-\underline{f}_i^+(s,Y_s^{+,i,0},Z_s^{+,i,0})+U_t^i,$$

and

$$N_t^i:=Z_t^{+,i,0}+V_t^i-\mathbf{1}_{\{Y_t^{+,i,0}+B^i(t)>S^i(t)\}}\Big(Z_t^{+,i,0}+V_t^i-\bar{V}_t^i\Big).$$

Note that $\underline{f}_i^+$ and $\underline{f}_i^-$ are uniformly Lipschitz continuous in (y,z^i). Thus, using the fact that $(Y^{+,i,0},Z^{+,i,0})$ and $(Y^{-,i,0},Z^{-,i,0})$ belong respectively to $\mathcal{S}^2\times\mathcal{M}^{d,2}$, yields that there is a constant $C>0$ such that

$$\mathbb{E}\Big\{\int_0^T\Big(|\underline{f}_i^+(t,Y_t^{+,i,0},Z_t^{+,i,0})|^2+|\underline{f}_i^-(t,Y_t^{-,i,0},Z_t^{-,i,0})|^2\Big)\,dt\Big\}\leq C. \quad (22)$$

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Moreover, in view of the above and assumption [**H5**], there exists a constant $C > 0$ such that

$$E\left[\int_0^T (|M_t^i|^2 + |N_t^i|^2)dt\right] \le C < +\infty. \tag{23}$$

Then, applying [10, Proposition 4.2] yields for all $t \le T$

$$0 \le dK_t^{-,i,0} \le \mathbf{1}_{\{Y_t^{-,i,0} = \Xi_t^i\}} \left[\underline{f}_i^-(s, Y_s^{-,i,0}, Z_s^{-,i,0}) + M_t^i\right]^+ dt, \tag{24}$$

which means that, $K^{-,i,0}$ is absolutely continuous w.r.t. t. Next, in the same spirit, we can show, thanks to [10, Proposition 4.2], for all $n > 0$ that the process $K^{-,i,n+1}$ is absolutely continuous w.r.t. t. Furthermore, we can obtain that: there exists a constant $C > 0$ such that for all $n \ge 0$ and $i \in \Lambda$,

$$E\left[\int_0^T \Big(\frac{dK^{-,i,n}}{dt}\Big)^2 dt\right] \le C. \tag{25}$$

Notice that, by combining [**H1**](ii) together with (15), (16), (20)we obtain that there is a constant $C > 0$ such that

$$\sup_{n\ge 0} \mathbb{E}\bigg\{ \int_0^T \Big| f_i^-(t, Y_t^{-,1,n}, \ldots, Y_t^{-,i-1,n}, Y_t^{-,i,n+1}, Y_t^{-,i+1,n}, \ldots, Y_t^{-,m,n}, \\ Z_t^{-,i,n+1})\Big|^2 \, dt\bigg\} \le C < +\infty. \tag{26}$$

Next, in view of estimates (20), (25) and (26), we deduce that there exists a subsequence along which all $((\frac{dK_t^{-,i,n+1}}{dt})_{0\le t\le T})_{n\ge 0}$, $((Z_t^{-,i,n+1})_{0\le t\le T})_{n\ge 0}$ and $((f_i^-(t, Y_t^{-,1,n}, \ldots, Y_t^{-,i-1,n}, Y_t^{-,i,n+1}, Y_t^{-,i+1,n}, \ldots, Y_t^{-,m,n}, Z_t^{-,i,n+1}))_{0\le t\le T})_{n\ge 0}$ converge weakly in their respective spaces $\mathcal{M}^{1,2}$, $\mathcal{M}^{d,2}$ and $\mathcal{M}^{1,2}$ to the processes $(k_t^{-,i})_{0\le t\le T}$, $(Z_t^{-,i})_{0\le t\le T}$ and $(\varphi^{-,i}(t))_{0\le t\le T}$.
Next, for any $n \ge 0$ and any stopping time τ we have

$$Y_\tau^{-,i,n+1} = Y_0^{-,i,n+1} + K_\tau^{-,i,n+1} + \int_0^\tau Z_s^{-,i,n+1} dB_s \\ - \int_0^\tau f_i^-(t, Y_s^{-,1,n}, \ldots, Y_s^{-,i-1,n}, Y_s^{-,i,n+1}, Y_s^{-,i+1,n}, \ldots, Y_s^{-,m,n}, Z_s^{-,i,n+1}) ds.$$

Taking the weak limits in each side and along this subsequence yields

$$Y_\tau^{-,i} = Y_0^{-,i} - \int_0^\tau \varphi^{-,i}(s) ds + \int_0^\tau k_s^{-,i} ds + \int_0^\tau Z_s^{-,i} dB_s, \quad \mathbb{P}\text{–a.s.}$$

Since the processes appearing in each side are optional, using the Optional Section Theorem (see e.g. [3], Chapter IV pp.220), it follows that

$$Y_t^{-,i} = Y_0^{-,i} - \int_0^t \varphi^{-,i}(s) ds + \int_0^t k_s^{-,i} ds + \int_0^t Z_s^{-,i} dB_s, \ \forall\, t \le T, \ \mathbb{P}\text{–a.s.} \tag{27}$$

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Therefore, the process $Y^{-,i}$ is continuous. Relying both on Dini's Theorem and on Lebesgue's dominated convergence one (17), we also get that

$$\lim_{n\to\infty} E\left[\sup_{0\le t\le T} |Y_t^{-,i,n} - Y_t^{-,i}|^2\right] = 0. \tag{28}$$

We will now focus on the continuity of the sequence of processes $(Y_t^{+,i})_{0\le t\le T}$, $\forall\ i \in \Lambda$. Actually, applying Peng's Monotone Limit Theorem (see [15]) yields that for every $i \in \Lambda$, the limit process $Y^{+,i}$ is càdlàg. Based on what has been already shown in previous steps, by mimicking the arguments of [15] we can easily show that there exist two processes $K^{+,i} \in \mathcal{K}_c^2$ and $Z^{+,i} \in \mathcal{M}^{d,2}$ such that $Y^{+,i}$ satisfies the first equation of RBSDE (S). Moreover, passing to the limit in the fifth inequality of RBSDE (11), implies that $Y_t^{+,i} \ge S_t^{+,i},\ t \in [0,T],\ i \in \Lambda$. Thus, for $i \in \Lambda$, $(Y^{+,i}, Z^{+,i}, K^{+,i})$ satisfies

$$\begin{cases} Y_t^{+,i} = \xi_i^+ + \displaystyle\int_t^T f_i^+(s, \overrightarrow{Y}_s^{+,i}, Z_s^{+,i})ds - \int_t^T Z_s^{+,i} dW_s + K_T^{+,i} - K_t^{+,i}, \\ Y_t^{+,i} \ge S_t^{+,i}. \end{cases} \tag{29}$$

It remains to prove the minimal boundary condition. Next, consider the following RBSDE whose solution exists thanks to [16]:

$$\begin{cases} \widetilde{Y}_t^{+,i} = \xi_i^+ + \displaystyle\int_t^T f_i^+(s, Y_s^{+,1}, \ldots, Y_s^{+,i-1}, \widetilde{Y}_s^{+,i}, Y_s^{+,i+1}, \ldots, Y_s^{+,m}, \widetilde{Z}_s^{+,i})\, ds \\ \qquad\qquad - \displaystyle\int_t^T \widetilde{Z}_s^{+,i} dW_s + \widetilde{K}_T^{+,i} - \widetilde{K}_t^{+,i}, \\ \widetilde{Y}_t^{+,i} \ge S_{t-}^{+,i}, \quad \text{and} \quad \displaystyle\int_0^T \left[\widetilde{Y}_{s^-}^{+,i} - S_{s-}^{+,i}\right] d\widetilde{K}_s^{+,i} = 0. \end{cases} \tag{30}$$

Note that RBSDEs (29) and (30) have the same lower barrier. In fact, since $\widetilde{Y}_t^{+,i}$ is the smallest f_i^+–supermartingale with lower barrier $S_t^{+,i}$, we have that for any $i \in \Lambda$, $\widetilde{Y}_t^{+,i} \le Y_t^{+,i}$ (see [16, Theorem 2.1]). On the other hand since for any $i \in \Lambda$ and $n \ge 1$, $Y_t^{+,i,n} \le Y_t^{+,i}$ and $Y_t^{-,i,n+1} \le Y_t^{-,i}$, applying the comparison theorem in view of **[H2]**(ii) yields that $Y_t^{+,i,n+1} \le \widetilde{Y}_t^{+,i}$, and then passing to the limit implies that $Y_t^{+,i} \le \widetilde{Y}_t^{+,i}$. Summing up we have that for any $i \in \Lambda$, $Y_t^{+,i} = \widetilde{Y}_t^{+,i}$. From the uniqueness of the Doob-Meyer decomposition, it follows that $Z_t^{+,i} = \widetilde{Z}_t^{+,i}$, $dt \times d\mathbb{P}$–a.s., and $K_t^{+,i} = \widetilde{K}_t^{+,i}$ for any $0 \le t \le T$, $\mathbb{P}$–a.s. Then, for $i \in \Lambda$, $(Y^{+,i}, Z^{+,i}, K^{+,i})$ satisfies RBSDE (5) but with the following minimal boundary condition

$$\int_0^T \left[Y_{s^-}^{+,i} - \max_{j\in\Lambda^{-i}} h_{i,j}(s, Y_{s-}^{+,j}) \vee \left(Y_s^{-,i} + C^i(s)\right)\right] dK_s^{+,i} = 0. \tag{31}$$

From the first equation of (29) and since the process $K_t^{+,i}$ is increasing, it follows that $\Delta Y_t^{+,i} = -\Delta K_t^{+,i} \le 0$. Assume that $\Delta Y_{t^*}^{+,i_1} < 0$, for some $(i_1, t^*) \in$

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$\Lambda \times [0,T]$. Thus $\Delta K^{+,i_1}_{t^*} > 0$. From the minimality condition (31), we have

$$Y^{+,i_1}_{t^*-} = \max_{j\in\Lambda^{-i_1}} h_{i_1,j}(t^*, Y^{+,j}_{t^*-}) \vee \Big(Y^{-,i_1}_{t^*} + C^{i_1}(t^*)\Big).$$

Let $i_2 \in \Lambda^{-i_1}$ be the optimal index for which the maximum is attained. Thus,

$$\begin{aligned} h_{i_1,i_2}(t^*, Y^{+,i_2}_{t^*-}) \vee (Y^{-,i_1}_{t^*} + C^{i_1}(t^*)) &= Y^{+,i_1}_{t^*-} \\ &> Y^{+,i_1}_{t^*} \\ &= h_{i_1,i_2}(t^*, Y^{+,i_2}_{t^*}) \vee (Y^{-,i_1}_{t^*} + C^{i_1}(t^*)). \end{aligned} \tag{32}$$

This obviously yields that $Y^{+,i_1}_{t^*-} = h_{i_1,i_2}(t^*, Y^{+,i_2}_{t^*-}) > h_{i_1,i_2}(t^*, Y^{+,i_2}_{t^*})$, and thus $\Delta Y^{+,i_2}_{t^*} < 0$. Repeating the above procedure we obtain for $i_k \in \Lambda^{-i_{k-1}}$

$$\Delta Y^{+,i_k}_{t^*} < 0, \quad \text{and} \quad Y^{+,i_k}_{t^*-} = h_{i_k,i_{k+1}}(t^*, Y^{+,i_{k+1}}_{t^*-}), \quad k = 2,\dots,m.$$

Since each i_k can take only values in Λ which is a finite set, then there must be a loop in Λ. we may assume w.l.o.g. that $i_{k+1} = i_1$ for some $k \geq 2$ noting again that the i_k's are mutually different i.e. for each k, $i_k \in \Lambda^{-i_{k-1}}$. Therefore, we have $Y^{+,i_1}_{t-} = h_{i_1,i_2}(t^*, Y^{+,i_2}_{t^*-}), \dots, Y^{+,i_{k-1}}_{t-} = h_{i_{k-1},i_k}(t^*, Y^{+,i_k}_{t^*-})$, and $Y^{+,i_k}_{t-} = h_{i_k,i_1}(t^*, Y^{+,i_1}_{t^*-})$. This contradicts assumption **[H2]**(iv). Consequently, $\Delta Y^{+,i}_t = \Delta K^{+,i}_t = 0,\ t \in [0,T],\ \forall i \in \Lambda$. Hence, the processes $Y^{+,i}$ and $K^{+,i}$, $i \in \Lambda$ are continuous.

Step 5: Identification of the limit

Next, we show that

$$\lim_{n\to\infty} \mathbb{E}\{\sup_{0\leq t\leq T} |Y^{+,i,n}_t - Y^{+,i}_t|^2 + |K^{+,i,n}_T - K^{+,i}_T|^2 + \int_0^T |Z^{+,i,n}_t - Z^{+,i}_t|^2 dt\} = 0.$$

Actually, since $Y^{+,i,n} \nearrow Y^{+,i}$ and $Y^{+,i}$ is continuous then relying both on Dini's Theorem and on Lebesgue's dominated convergence one (17), we get that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |Y^{+,i,n}_t - Y^{+,i}_t|^2\right] = 0. \tag{33}$$

Further, we can easily show by applying Itô's formula to $|Y^{\pm,i,n}_t - Y^{\pm,i,p}_t|^2$ $(n,p \geq 0)$ and using standard arguments (see e.g. [10]) that

$$\lim_{n\to\infty} \mathbb{E}[\int_0^T |Z^{\pm,i,n}_t - Z^{\pm,i}_t|^2 dt + |K^{+,i,n}_T - K^{+,i}_T|^2 + |K^{-,i,n}_T - \int_0^T k^{-,i}_s ds|^2] = 0.$$

From this, and **[H1]**(ii) combined with (27), (28) it holds that

$$\varphi^-_i(t) = f^-_i(t, \overrightarrow{Y}^-_t, Z^{-,i}_t), \quad 0 \leq t \leq T.$$

Next, passing to the limit in the second inequality of RBSDE (11), yields that $Y^{-,i}_t \leq S^{-,i}_t,\ t \in [0,T],\ i \in \Lambda$. Furthermore, thanks to the weak convergence of

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$((\frac{dK_t^{-,i,n}}{dt})_{0\leq t\leq T})_{n\geq 1}$ to the process $k^{-,i}$ and the strong convergences (28) and (33), we deduce that

$$0 = \int_0^T \left[S^i(t) \wedge \left(Y_t^{+,i,n} + B^i(t)\right) - Y_t^{-,i,n+1}\right] dK_t^{-,i,n+1}$$
$$\longrightarrow \int_0^T \left[S_t^{-,i} - Y_t^{-,i}\right] k_t^{-,i} dt = 0, \quad \text{as } n \to +\infty.$$

In fact, this implies that $(Y^{-,i}, Z^{-,i}, K^{-,i} := \int_0^\cdot k_s^{-,i} ds)$ is a solution to the second part of RBSDE (S). Summing up $(Y^{\pm,i}, Z^{\pm,i}, K^{\pm,i})$ is a solution of RBSDE (S). Finally, it remains to show that this solution is the minimal one.

Step 6: Minimality of the solution of RBSDE (S)

Let $(\bar{Y}^{+,i}, \bar{Z}^{+,i}, \bar{K}^{+,i}, \bar{Y}^{-,i}, \bar{Z}^{-,i}, \bar{K}^{-,i})$ be another solution of RBSDE (S). Since $Y^{+,i,n} \leq Y^{+,i}$ and $Y^{-,i,n} \leq Y^{-,i}$, for all $n \geq 0$, and thanks to the monotonicity of $h_{i,j}$ applying the comparison theorem yields that for each $i \in \Lambda$: $Y^{+,i,n} \leq \bar{Y}^{+,i}$ and $Y^{-,i,n} \leq \bar{Y}^{-,i}$, for all $n \geq 0$. Passing to the limit when $n \to \infty$ implies that for each $i \in \Lambda$: $Y^{+,i} \leq \bar{Y}^{+,i}$ and $Y^{-,i} \leq \bar{Y}^{-,i}$, which is the desired result. This ends the proof of Theorem 1.

4 Conclusion and perspectives

In this paper we have proved the existence of a continuous minimal solution to RBSDE (S) which is arising from BSOSPs. Let us comment on a possible generalization of the results obtained in this paper. Actually, the full balance sheet case is still an open problem and constitutes a challenge. By the full balance sheet case we mean that we consider the two sides of the balance sheet. Indeed, in this case the expected cost in mode i, $Y^{-,i}$ should rise above the following barrier

$$\min_{j \in \Lambda^{-i}} l_{i,j}(t, Y_t^{-,j}) \wedge \left(Y_t^{+,i} + B^i(t)\right), \tag{34}$$

instead of $S_t^{-,i}$ where $l_{i,j}$ is a real nonlinear random function satisfying **[H2]**, except for **[H2]**–(iii) which is replaced by $l_{i,j}(t,y) \geq y$. We want to stress out that, the new assumption **[H2]** is satisfied when $l_{i,j}$ takes the particular form $l_{i,j}(\cdot, y) = y + g_{i,j}$ where $g_{i,j}$ is the switching cost from mode i to mode j.

A full BSOSP amounts to establishing existence of a continuous solution to the system of RBSDEs (S), but with the upper barrier (34) for $Y^{-,i}, \forall i \in \Lambda$. Note that, as in the proof of Theorem 1 (**Step 4**) the absolute continuity of the process $K_t^{-,i}$ w.r.t. t will play a primordial role to derive convergence of the corresponding approximating sequence. To do so we need to use the Itô–Tanaka formula (see **Step 4**), which makes it difficult to solve the system of RBSDEs (S) for the full balance sheet case. Note that even in the case when the functions $h_{i,j}$ and $l_{i,j}$ take the particular forms respectively $h_{i,j}(\cdot, y) = y - g_{i,j}$ and $l_{i,j}(\cdot, y) = y + g_{i,j}$ let alone the general case, the question of existence of solutions to the corresponding system of RBSDEs (S) for the full balance sheet case, is still open. This issue was discussed in [5] and in [6] in the mean field case.

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