Path-dependent interest rate option pricing with jumps and stochastic intensities

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Abstract. We derive numerical series representations for option prices on interest rate index for affine jump-diffusion models in a stochastic jump intensity framework with an adaptation of the Fourier-cosine series expansions method, focusing on the European vanilla derivatives. We give the price for nine different Ornstein-Uhlenbeck models enhanced with different jump size distributions. The option prices are accurately and efficiently approximated by solving the corresponding set ordinary differential equations and parsimoniously truncating the Fourier series.

Keywords: Interest rate · Option Pricing · COS method · AJD models.

1 Introduction

The main interest rate option traded in B3 is the IDI (interfinancial deposit index) Option, which is of European type and cash settled at maturity. The IDI is an index that accumulates from an initial value according to the daily calculated effective market rate DI. Analytical solutions for pricing IDI options can be found for the short rate given by the Vasicek model [14] and the CIR model [3]. [1] and [12] developed the closed-form with the known Hull-White model [11]. [4] implemented the HJM model to price IDI options. The problem was numerically solved via a finite difference method in [13] and via an alternative generic method in [2]. Another result was developed by [10], where the model is sensitive to changes in monetary policy. A discussion about this type of path-dependent option is found in two books ([6] and [7]).

The COS method is a Fourier inversion method introduced by [9]. It is a procedure to calculate probability density functions and expectations via cosine series. It originally relies on the explicit formula for the characteristic function of the state variable. This paper accomplishes an original contribution for the financial literature via devising a way to apply the COS method to affine jump-diffusion (AJD) models which do not pursue analytical solutions for the Ric-

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cati equations. More specifically, we develop option prices for AJD models with stochastic jump intensities, which can capture different market stress scenarios.

The paper is organized as follows: in Section 2 we review the Fourier-cosine expansion method to recover density functions and calculate the price of financial derivatives. In Section 3 we present the pricing problem and show the ordinary differential equations which give the solution of the AJD characteristic functions of the interest rate processes. In Section 4 we develop the coefficients related to the payoff functions. In Section 5 we present the probability distributions of the models and the convergence of the prices.

2 Fourier Series Method

An interesting, fast and accurate pricing method based on Fourier series expansions was recently proposed by [9] for options on stocks, with good potential to be shaped for other derivatives. Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be an integrable function. A change of variable $\xi = \pi \frac{x-a}{b-a}$ is considered in order to have an even function. Then, the Fourier-cosine series expansion of f in the interval $[a, b]$ is

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j cos\left(j\pi \frac{x-a}{b-a}\right), \tag{1}$$

where

$$a_j = \frac{2}{b-a} \int_a^b f\left(\pi \frac{x-a}{b-a}\right) cos\left(j\pi \frac{x-a}{b-a}\right) dx, \quad j \geq 0. \tag{2}$$

Let us assume that $f \in L^1(\mathbb{R})$ and that we explicitly know the Fourier transform of f. The approximation in the interval $[a, b]$ of the coefficients of the Fourier-cosine expansion of f is

$$a_j = \frac{2}{b-a} \int_a^b f(\xi) \Re\left(e^{ij\pi \frac{\xi-a}{b-a}}\right) d\xi \approx \frac{2}{b-a} \Re\left(e^{-ij\pi \frac{a}{b-a}} \hat{f}\left(\frac{j\pi}{b-a}\right)\right) \triangleq A_j. \tag{3}$$

The approximation of f is given by the following Fourier-cosine series:

$$f(x) \approx \frac{A_0}{2} + \sum_{j=1}^{n} A_j cos\left(j\pi \frac{x-a}{b-a}\right), \quad x \in [a, b], \tag{4}$$

for an appropriated chosen n.

Let X_T be a random variable with probability density function f_{X_T} with known characteristic function $\hat{f}(\omega) = \int_{\mathbb{R}} e^{ix\omega} f(x) dx \approx \int_a^b e^{ix\omega} f(x) dx$. The price of a European call option $C(t, T)$ with payoff function g and strike K is

$$C(t,T) \approx \frac{A_0}{2} \int_a^b g(X_T) dx + \sum_{j=1}^{n} A_j \int_a^b g(X_T) cos\left(j\pi \frac{x-a}{b-a}\right) dx. \tag{5}$$

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Hence, the series approximation of the option price is given by

$$C(t,T) = \mathbb{E}[g(X_T)|X_t] \approx \frac{A_0 B_0}{2} + \sum_{j=1}^{n} A_j B_j, \tag{6}$$

where the A_j coefficients are given by (3) and

$$B_j = \int_a^b g(X_T) cos\left(j\pi \frac{x-a}{b-a}\right) dx, \quad \text{for } \; j = 0, 1, ..., n. \tag{7}$$

The choices of the integration limits (a, b) for the approximation were proposed in [9]. The scenario we are dealing with is that where no closed-form exists for the characteristic function. Then, the limits are achieved by numerically differentiate the cumulant function of the model.

3 IDI options with affine jump-diffusion models

We assume an interest rate market with underlying probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ where $\mathbb{P}$ is the risk neutral measure.

Let r_t be the spot continuously compounding interest rate given by

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dB_t + JdN(\lambda t), \tag{8}$$

where $\mu(r_t, t)$ is the mean, $\sigma(r_t, t)$ is the volatility and B_t the standard Wiener process. N is a pure jump process with positive intensity $\lambda = \lambda_0 + \lambda_1 r_t$ and jump amplitudes J, which are i.i.d. and independent of B_t.

According to the B3 protocols, the DI rate is the average of the interbank rate of a one-day-period, calculated daily and expressed as the effective rate per annum. So, the ID index accumulates discretely, according to

$$y_t = y_0 \prod_{k=1}^{t} (1 + DI_k)^{\frac{1}{252}}, \tag{9}$$

where k denotes the end of day and DI_k assigns the corresponding DI rate. More details about the DI index is found in [7].

If $r_t = \ln(1 + DI_t)$, the index can be represented by the following continuous compounding expression $y_t = y_0 e^{\int_0^t r_s ds}$, where r_t is given by (8). The price for the option with strike K and maturity in t is given by

$$C_0 = \mathbb{E}\left[\max\left(y_0 - Ke^{-\int_0^t r_s ds}, 0\right) \Big| \mathcal{F}_0\right]. \tag{10}$$

From now on we benefit from the procedure found in [8] which focus in obtaining characteristic functions of affine jump-diffusion (AJD) models. We find a function of the solution of the AJD model from which a characteristic function should be obtained.

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Theorem 1. *The characteristic function of the integrated process* $x_t = \int_0^t r_s ds$ *where* r_s *is given by an AJD model of the form* (8) *is*

$$\hat{f}(x_t, iu) = \mathbb{E}\left[\exp\left(iux_t\right)|r_0\right] = \exp(\alpha(t) + \beta(t)r_0), \tag{11}$$

where

$$\alpha'(s) = \beta(s)\kappa + \lambda_0 \left[\mathbb{E}\left(e^{\beta(s)J_i}\right) - 1\right], \tag{12}$$

$$\beta'(s) = \beta(s)\theta + \frac{1}{2}\beta(s)^2\sigma^2 - iu + \lambda_1 \left[\mathbb{E}\left(e^{\beta(s)J_i}\right) - 1\right], \tag{13}$$

with boundary conditions $\alpha(0) = \beta(0) = 0$.

Proof. See [8].

The variable λ_0 multiplying the expectation in (12) gives the constant intensity of the jumps and the variable λ_1 multiplying the expectation in (13) gives the slope when the intensity of the jumps is stochastic. Closed-form solutions for $\left[\mathbb{E}\left(e^{\beta(s)J_i}\right) - 1\right]$ are found in [5] for exponential, normal and gamma jumps.

When dealing with stochastic intensity for the jumps, closed-form solutions for the Riccati equations do not exist. Solving (12) and (13) numerically with the Runge-Kutta algorithm gives the characteristic function we use in A_j coefficients of the series (6). Finite difference method is then used to differentiate the cumulant-generating function in order to prescribe the integration limits.

In this paper, we deal with a variety of Ornstein-Uhlenbeck processes. The Vasicek model without jumps, the Vasicek model with exponential positive and negative jumps, the Vasicek model with stochastic positive and negative exponential jumps, the Vasicek model with normal and stochastic normal jumps and the Vasicek model with gamma and stochastic gamma jumps.

4 IDI Option pricing with the COS method

The characteristic function of the random variable $\int_0^t r_s ds$ enters in the A_j coefficients in equation (6). Whence, this suffices for calculating the A_j. So we only have to calculate the corresponding B_j coefficients in order to price the IDI option. We consider from now on the vanilla call option case as shown in equation (10). To the best of authors' knowledge, this is the first paper to study this class of models for path-dependent interest rate derivatives.

Theorem 2. *The* B_j *coefficients for vanilla IDI call options are given by*

$$B_0 = \int_{-\ln\left(\frac{y_0}{k}\right)}^{b} y_0 - k\mathrm{e}^{-x} dx = y_0 \left(\ln\left(\frac{y_0}{k}\right) + b - 1\right) + \mathrm{e}^{-b}k, \tag{14}$$

and

$$B_j = \int_{-\ln\left(\frac{y_0}{k}\right)}^{b} \left(y_0 - k\mathrm{e}^{-x}\right) \cos\left(\frac{\pi j\left(x-a\right)}{b-a}\right) dx \tag{15}$$

Proof. The vanilla IDI call option is given by (10). Integrating it according to equation (7) gives Theorem 2.

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5 Numerical results

In the following panels we exhibit the probability density functions of the integrated process of each model under study and the corresponding convergence analysis of the IDI call option prices. In figure 1 we show the Vasicek model with exponential jumps. The model is enhanced with positive and negative jumps. In figure 2 we show the Vasicek model with normal jumps. In figure 3 we show the the Vasicek model with gamma jumps. All above models were enhanced with constant and stochastic intensities. The base parameters for the figures are: $r_0 = 0.1$, $\kappa = 0.25$, $\theta = 0.1$, $\sigma = 0.04$, $\lambda_0 = 1$, $\lambda_1 = 10$, $\eta = 0.01$, $\mu = 0$, $\Sigma = 0.015$, $p = 1.5$, , $T = 5$, $y_0 = 100000$ and $K = 165000$.

We highlight that the numerical COS method converges with few terms in the Fourier series, namely around fifteen terms. It takes a half of second to calculate the price performing the Runge-Kutta method inside each term of the series and achieving an error of the order of 10^{-3}. The computer used for all experiments has an Intel Core i5 CPU, 2.53GHz. The code was written in MATLAB 7.8.

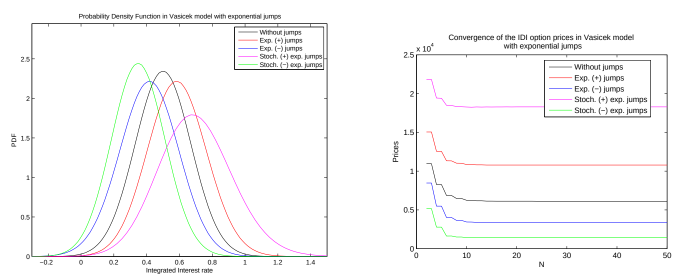

Fig. 1: (Left) Probability density functions for Vasicek model with exponential jumps. (Right) Convergence analysis for Vasicek model with exponential jumps.

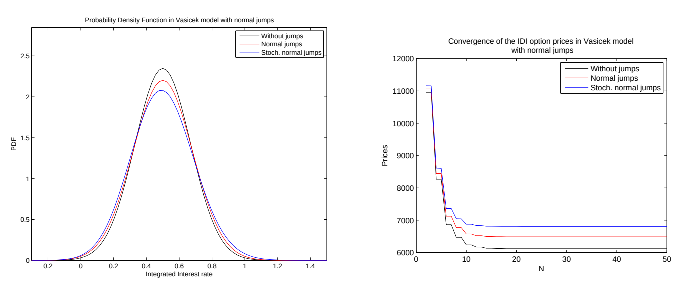

Fig. 2: (Left) Probability density functions for Vasicek model with normal jumps. (Right) Convergence analysis for Vasicek model with normal jumps.

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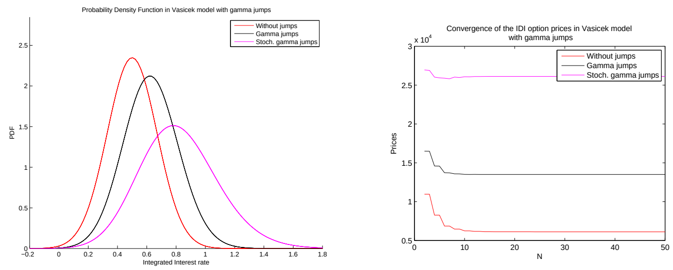

Fig. 3: (Left) Probability density functions for Vasicek model with gamma jumps. (Right) Convergence analysis for Vasicek model with gamma jumps.

6 Conclusion

We extended the range of application of the COS method to interest rate derivatives contracts. We benefited from the procedure found in [8] to obtain characteristic functions related to affine jump-diffusion (AJD) models. In this paper, we are not restricted to explicit solutions of the characteristic function. We provided the probability densities for interest rate models with stochastic intensities and the corresponding prices for a financial product found in the Brazilian market, the IDI option. We devised a path-dependent function that corresponds to the integral of the interest rate process, from which the numerical values for the associated characteristic function was calculated via Runge-Kutta method. We show that the prices converge fastly regarding the number of terms of the Fourier series. To the best of our knowledge, this is the fastest method to numerically calculate the price of a path-dependent interest rate option.

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